

Exact solutions of closed string theory

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Abstract

We review explicitly known exact $D = 4$ solutions with Minkowski signature in closed bosonic string theory. Classical string solutions with space-time interpretation are represented by conformal sigma models. Two large (intersecting) classes of solutions are described by gauged WZW models and ‘chiral null models’ (models with conserved chiral null current). The latter class includes plane-wave type backgrounds (admitting a covariantly constant null Killing vector) and backgrounds with two null Killing vectors (e.g., fundamental string solution). $D > 4$ chiral null models describe some exact $D = 4$ solutions with electromagnetic fields, for example, extreme electric black holes, charged fundamental strings and their generalisations. In addition, there exists a class of conformal models representing axially symmetric stationary magnetic flux tube backgrounds (including, in particular, the dilatonic Melvin solution). In contrast to spherically symmetric chiral null models for which the corresponding conformal field theory is not known explicitly, the magnetic flux tube models (together with some non-semisimple WZW models) are among the first examples of solvable unitary conformal string models with non-trivial $D = 4$ curved space-time interpretation. For these models one is able to express the quantum hamiltonian in terms of free fields and to find explicitly the physical spectrum and string partition function.

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1. Introduction

String theory is a remarkable extension of General Relativity which is consistent at the quantum level. One of the important problems is to study the set of exact classical solutions to string equations of motion. This may clarify its formal aspects but also may be relevant for understanding the implications of string theory for cosmology and black holes (assuming that in certain regions, e.g., at small times/scales, the effective string coupling is small so that perturbative and non-perturbative corrections to string equations of motion can be ignored). To be able to address issues of singularities and strong field behavior one should determine not just solutions of the leading-order low-energy string effective equations derived under the assumption of small field gradients ($|\alpha'R| \ll 1$, etc.) but solutions that are exact to all orders in α' (i.e., the corresponding exact conformal σ -models), and, ultimately, how first-quantized string modes propagate in a given background (i.e., the underlying conformal field theories).

1.1. Sigma model approach

In addition to its dependence on extra ‘massless’ fields (antisymmetric tensor and dilaton in bosonic string case) string theory ‘massless’ effective action contains terms of all orders in derivatives multiplied by powers of α' . When this effective action is derived by starting from the string S-matrix [1] its full structure is hard to determine explicitly and the problem of finding its extrema exactly to all orders in α' looks hopeless. A remarkable possibility to find exact backgrounds without knowing explicit form of the effective action is based on the correspondence between classical string solutions and conformal σ -models.

Let us briefly recall the basic features of the σ -model approach. The recognition of the role of σ -models in string theory was prompted by the covariant path integral representation for the string-theory scattering amplitudes [2] and by studies of σ -models as geometrical 2-dimensional quantum field theories [3,4]. It was observed [5] that the generating functional for the string scattering amplitudes can be interpreted as a partition function Z of a σ -model with couplings being the string space-time fields. That suggested a relation between Z and the string effective action S leading to the expression for S which, in contrast to the one in the S-matrix approach, was manifestly covariant and non-perturbative in number of external field ‘legs’. An independent development was the realisation that a necessary condition on a string solution is that the σ -model describing string propagation in this background should be conformally invariant [6]. It was checked [7,8] that the leading order conditions of conformal invariance of a σ -model do indeed correspond to extrema of the leading order term in the effective action. The proportionality between the Weyl anomaly coefficients (‘ $\bar{\beta}$ -functions’) [9] and derivatives of the effective action was suggested to hold also off shell [10,11] and the field redefinition (scheme choice) ambiguity in the effective action (‘ $\bar{\beta}$ -functions’) was understood [11,12]. That made possible to carry out the non-trivial test of equivalence between the σ -model ‘ $\bar{\beta}$ -functions’ and the effective action derived from the string S-matrix at the next to leading order [13,14]. The observation that a renormalisation of the σ -model corresponds to a subtraction of massless poles in the string scattering amplitudes [15] led to the expression for the (S-matrix derived) closed string effective action S in terms of the renormalised derivative of the σ -model partition function Z over the 2d cutoff [16] (correcting the original ansatz $S \sim Z$ of [5] which is valid only in the open string theory). This representation provided a basis for a proof that the extrema of the effective action indeed correspond to the Weyl invariant σ -models ([16,17,18] and refs. there). For reviews of σ -model approach see, e.g., [18,19].

To summarize, in the ‘first-quantized’ (or ‘ σ -model’) approach a string background is represented by a 2-dimensional σ -model. The fundamental feature of the closed string

theory is that a large-scale classical approximation to the geometry is thus described not only by the metric but also by the antisymmetric tensor and the scalar dilaton field,

$$I = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{\gamma} \left[\partial_m x^\mu \partial^m x^\nu G_{\mu\nu}(x) + i\epsilon^{mn} \partial_m x^\mu \partial_n x^\nu B_{\mu\nu}(x) + \alpha' R^{(2)} \phi(x) \right]. \quad (1.1)$$

γ is a world sheet metric which decouples once the background fields satisfy the classical string field equations.¹

In the classical (leading order in string coupling) approximation the string field equations are the conditions of conformal (Weyl) invariance of σ -model (1.1) which are equivalent to the conditions of stationary of the tree-level effective action S

$$\bar{\beta}^i = \kappa^{ij}(\varphi) \frac{\delta S}{\delta \varphi^i}, \quad \varphi^i = (G_{\mu\nu}, B_{\mu\nu}, \phi), \quad (1.2)$$

where κ^{ij} is (in general) non-degenerate and

$$S = \int d^D x \sqrt{G} e^{-2\phi} \left[(D-26) - \frac{3}{2} \alpha' (R + 4\nabla^2 \phi - 4(\nabla \phi)^2 - \frac{1}{12} (H_{\mu\nu\lambda})^2) + O(\alpha') \right], \quad (1.3)$$

reproduces the ‘massless’ sector of the string S-matrix. Under a special choice of definitions of the fields (i.e. in a special ‘scheme’) S admits also the following σ -model representations

$$S = \int d^D x \sqrt{G} e^{-2\phi} \tilde{\beta}^\phi, \quad S = \left(\frac{\partial Z}{\partial t} \right)_{t=1}, \quad (1.4)$$

where $\tilde{\beta}^\phi$ is the σ -model Weyl anomaly coefficient $\tilde{\beta}^\phi = \bar{\beta}^\phi - \frac{1}{4} G^{\mu\nu} \bar{\beta}_{\mu\nu}^G$, Z is the σ -model partition function on a 2-sphere and t is the logarithm of the renormalisation parameter. The tree-level string dynamics is thus encoded in the quantum 2d σ -model with the classical string solutions corresponding to conformal 2d σ -models. It should be noted that this correspondence may break down for certain singular solutions: since κ^{ij} in (1.2) is field-dependent it may vanish at the points where the metric (or dilaton) is singular. As a result, a background which corresponds to a conformal σ -model may solve the string effective equations only in the presence of additional sources (see [20] and Section 5 below).

1.2. Conformal models as string solutions

The remarkable correspondence between the stationary points of the effective action and conformally invariant σ -models makes it possible to determine the exact solutions of the effective field equations which are not even known in an explicit form. All one needs to do is to find a conformal σ -model (e.g., proving that all α' corrections are absent in a special scheme). It is natural from string theory point of view to describe a string background

¹ The classical string follows the ‘geodesic’ equations corresponding to the action (1.1). One of the consequences of the extended nature of strings is that while a point-like string mode follows the geodesics of the metric, other string modes feel also the antisymmetric tensor background. First-quantised test string also feels the dilaton background [5].

not by a collection of space-time fields but directly by the σ -model itself. This has an important consequence: since 2d quantum σ -models may be related by certain equivalence transformations (e.g., world-sheet / target space duality), differently looking collections of background fields may actually represent the same string solution. More generally, one may define a string solution as the 2d conformal field theory (CFT) corresponding to a given conformal σ -model. CFT contains, of course, much more information than just one set of local background fields parametrising a conformal σ -model. CFT gives an adequate interpretation of a given string solution: it describes, in particular, how all modes of a first-quantised string propagate in this background. It should be emphasized, at the same time, that if one is interested in string solutions which should have a space-time interpretation, the knowledge of CFT at an algebraic level only is not sufficient: one should be able to relate it to a lagrangian field theory, i.e. there should exist weak field regions where a conformal σ -model description is adequate.

Thus, there are several problems one is to solve in order to give a string-theoretic description of a particular string background:

(1) First, one needs to specify some basic long-distance properties of a solution one is looking for. This may involve finding a solution of the leading-order string effective equations.

(2) Second, one is to promote this leading-order solution to an exact (all order in α') string solution by identifying a conformal σ -model to which it corresponds in the weak field (or $\alpha' \rightarrow 0$) approximation. It may happen that this conformal σ -model is the same as the leading-order one (i.e. its couplings are not modified by α' corrections) but in general the background fields are expected to be given by power series in α' .

(3) Finally, one is to define a $2d$ conformal field theory associated with the conformal σ -model and to solve it, determining the spectrum of string excitations on a given background and their scattering amplitudes. In particular, CFT encodes information about equations for all string modes in a given background.

The last step is, of course, the most difficult one. Unfortunately, only very few exact string solutions have known CFT which corresponds to them and only in few cases this CFT is solvable. Given a conformal σ -model (and thus exact expressions for the massless background fields) but having no explicit information about the underlying CFT, it is, in general, not possible to determine the exact form of the linearised equations for the tachyon and other string excitations since they may receive α' -corrections. As a result, one is unable, for example, to give a definite answer about (non)singularity of a string background as probed by test first-quantised strings.

1.3. *Classes of exact solutions*

This paper is devoted to a discussion of exact solutions of (bosonic) string theory, mainly at the σ -model level. While the solutions of the leading-order effective equations are numerous and straightforward to find, little is known about their exact counterparts. For example, the exact form of the string analog of the Schwarzschild solution remains unknown. It turns out that there exist a subset of solutions (and these are, in fact, the only exact solutions known at present) which are not modified by α' -corrections in a special scheme, i.e. retain their leading-order form.² It seems remarkable that this can happen at all, given that string effective action is an infinite series of terms of all orders in α' . The reason for that, of course, is that string effective action is special, being related to σ -model conformal anomaly coefficients by (1.2). Thus, given a conformal σ -model (e.g.,

² This does not mean, however, that the corresponding CFT Hamiltonian and thus equations for other string modes (e.g. tachyon) do not contain non-trivial α' -corrections.

$(4, n)$, $n \geq 1$ supersymmetric one or a special ‘chiral null’ bosonic one) one gets a string solution which is not modified by α' -corrections.

As was mentioned above, to obtain exact solutions in string theory it is rather hopeless to start with the field equations expressed as a power series in α' , and try to solve them explicitly. As was demonstrated in [21,20,22], one can find exact solutions by starting directly with some specific σ -models and explicitly proving that they are conformal. This σ -model approach leads to new solutions which look quite complicated and hard to find by explicitly solving the string effective equations even at the leading order. Thus by choosing an ansatz for the string world sheet action which yields simple equations for the σ -model β -functions, one can easily find new solutions of even the leading order equations.

There are several approaches that were used to construct exact string solutions:

(i) Start with a particular leading-order solution and show that α' -corrections are absent (or take some explicitly known form) due to some special properties of this background. Such are some ‘plane wave’-type backgrounds, or, more generally, certain backgrounds with a covariantly constant Killing vector, see, e.g., [23,24,25,26,27,28,29,21].

(ii) Start with a known lagrangian CFT and represent it as a conformal σ -model. Essentially the only known example of such construction is based on gauged WZW models, see, e.g., [30,31,32,33].

(iii) Start directly with a leading-order σ -model path integral and prove that there exists such a definition of the σ -model couplings (i.e. a ‘renormalisation scheme’) for which this σ -model is conformal to all loop orders. This strategy works for ‘ F -models’ [34,21], their ‘chiral null model’ generalisations [20] and the magnetic flux tube models [22]. A particular example of F -model is the ‘fundamental string’ solution [35]. Chiral null models describe also travelling waves along the fundamental string, charged fundamental strings and some electromagnetic backgrounds. Special $D = 5$ chiral null models can be interpreted (in a Kaluza-Klein way) as extreme $D = 4$ charged black holes [36].

The above three classes of exact solutions have common subsets. For example, a subclass of plane waves can be identified with gauged WZW models based on non-semisimple groups [37,38,39,40,41,42,43]. Also, a different subclass of backgrounds with a covariantly constant null Killing vector (called ‘ K -models’ in [21]) are dual to F -models. Finally, a subclass of F -models can be interpreted as special G/H gauged WZW models with G being a maximally non-compact real group and H being a nilpotent subgroup [34]; similar interpretation is true for a generic $D = 3$ F -model [21].

Conformal field theories corresponding to those of chiral null models which cannot be identified with gauged WZW models seem hard to solve explicitly. One particular approach one may follow in order to construct exact solutions described by conformal σ -models which are solvable as string theories (i.e. as CFT’s) was suggested in [22]. It was demonstrated there (on the example of the magnetic flux tube models) that the world-sheet duality transformations relating complicated σ -models to simpler ones are a useful tool for finding conformal σ -models with *solvable* CFT’s behind them. For example, starting with a dimension $D \geq d+n$ flat model with a number d of periodic coordinates and making formal $O(d+n, d+n; R)$ world-sheet duality transformations (see e.g. [44,45,46]) with *continuous* parameters one obtains new *inequivalent* conformal theories (with $O(d, d; Z)$ dualities as symmetries), corresponding to complicated space-time backgrounds which solve the string effective equations.

The backgrounds we shall consider below are exact solutions of both bosonic and superstring (and heterotic string) theories.³ In addition, there are special exact solutions of superstring theory only (corresponding to σ -models with extended $(4, n)$ world-sheet

³ For reviews of some superstring solutions see, e.g., [47,48,49] and references there.

and partial space-time supersymmetry). Even having in mind superstring applications, it is still important to understand mechanisms leading to exactness of particular solutions which may not be directly based on extended supersymmetry (some physically interesting solutions may have broken space-time supersymmetry).⁴

1.4. Outline

In Section 2 we shall briefly discuss the exact solutions corresponding to gauged WZW models, pointing out the existence of a ‘leading-order’ scheme in which the ‘massless’ background fields are α' -independent. We shall also mention some known solutions in $D = 4$ dimensions.

Gauged WZW models seem not to be sufficient to describe physically interesting $D \geq 4$ string backgrounds. In Section 3 we shall make some general remarks about another class of solutions with Minkowski signature which are described by σ -models with conserved chiral null currents. Particular subclasses of such backgrounds will be the topics of Sections 4-7.

Section 4 will be devoted to solutions with a covariantly constant null Killing vector. We shall start with simplest plane wave backgrounds (some of which will be related to gauged WZW models based on non-semisimple groups) and discuss several generalisations, e.g., a ‘hybrid’ $D = 4$ model with the transverse part represented by the euclidean $D = 2$ background.

Exact solutions with two null Killing vectors (F -models) will be presented in Section 5. In particular, we shall consider F -model with a flat transverse part which describes the fundamental string background and its generalisations. We shall also mention a subclass of F -models (with $D = 4$ and Minkowski signature) which correspond to specific nilpotently gauged WZW models and thus admit direct CFT interpretation.

Chiral null models generalising both the plane-wave models and F -models will be introduced in Section 6. In Section 7 we shall consider charged $D = 4$ backgrounds which are described by $D > 4$ chiral null models. They include, in particular, extreme electric black holes and charged fundamental strings. We shall explain the procedure of Kaluza-Klein-type reduction and review the known extreme magnetic and electric black holes which are exact (super)string solutions to all orders in α' . We shall also consider more general electromagnetic backgrounds (including IWP and Taub-NUT type ones), as well as a special ‘constant magnetic field’ solution which has no analog in Einstein-Maxwell theory.

In Section 8 we shall study a special $D \geq 5$ conformal σ -model which does not belong to the gauged WZW or chiral null model classes. Its $D = 5$ version (with one of the coordinates being a compact Kaluza-Klein one) describes $D = 4$ stationary axisymmetric magnetic flux tube backgrounds generalising both the dilatonic Melvin and the constant magnetic field solutions.

This model provides a remarkable example of a non-trivial curved space-time solution with an explicitly solvable conformal string theory corresponding to it. Its solution [51] will be reviewed in Section 9 where we shall first explain the relation of the magnetic flux tube

⁴ In particular, in claims about finiteness of supersymmetric σ -models it is a special holonomy of the bosonic background that often plays the crucial role. For example, it was pointed out in [50] that since the σ -model on a Calabi-Yau space has a special holonomy it thus has an extra infinite-dimensional non-linear classical symmetry which (if it were not anomalous at the quantum level) would rule out all higher-loop corrections to the β -function. In the case of the bosonic chiral null models the analogous symmetry is linear and is the affine symmetry generated by the chiral null current.

model to a flat space model via angular duality. As a result, it will be possible to solve the classical string equations explicitly, expressing the string coordinates in terms of free fields satisfying ‘twisted’ boundary conditions. After straightforward operator quantisation one then determines the quantum Hamiltonian, spectrum of states and partition function, in direct analogy with how this is done in simpler models like closed string on a torus or an orbifold, or open string in a constant magnetic field.

In Section 10 we shall make some concluding remarks.

2. Exact solutions corresponding to gauged WZW models

In this section we shall briefly discuss exact string solutions described by gauged WZW models (for more detailed reviews of different aspects of these solutions see, e.g., [52,53,54]). An important property of these solutions is that the corresponding CFT is, in principle, known explicitly. At the same time, it should be noted that most of the σ -model backgrounds obtained from gauged WZW models do not have non-abelian isometries and thus do not seem to be relevant for description of physically interesting spherically symmetric $D = 4$ cosmological and black hole backgrounds.

The general bosonic σ -model describing string propagation in a ‘massless’ background is given by (1.1), i.e., in conformal gauge,

$$L = (G_{\mu\nu} + B_{\mu\nu})(x) \partial x^\mu \bar{\partial} x^\nu + \mathcal{R}\phi(x) , \quad (2.1)$$

$$I = \frac{1}{\pi\alpha'} \int d^2z L , \quad \mathcal{R} \equiv \frac{1}{4}\alpha' \sqrt{\gamma} R^{(2)} ,$$

where $G_{\mu\nu}$ is the metric, $B_{\mu\nu}$ is the antisymmetric tensor and ϕ is the dilaton. The action of the ungauged WZW model for a group G [4]

$$I = kI_0 , \quad I_0 \equiv \frac{1}{2\pi} \int d^2z \operatorname{Tr} (\partial g^{-1} \bar{\partial} g) + \frac{i}{12\pi} \int d^3z \operatorname{Tr} (g^{-1} dg)^3 , \quad (2.2)$$

can be put into the general σ -model form (2.1) by introducing the coordinates on the group manifold. Then $G_{\mu\nu}$ is the group space metric, $H_{\mu\nu\lambda} = 3\partial_{[\mu} B_{\nu\lambda]}$ is the parallelising torsion and dilaton $\phi = \phi_0 = \text{const}$ (and $k = 1/\alpha'$). In a special scheme (where the β -function of the general model (2.1) is proportional to the generalised curvature) the resulting σ -model is finite at each order of α' -perturbation theory [55,56] and corresponds to the well-known current algebra CFT [4].

The action of the gauged WZW model [57]

$$I(g, A) = kI_0(g) + \frac{k}{\pi} \int d^2z \operatorname{Tr} (-A \bar{\partial} g g^{-1} + \bar{A} g^{-1} \partial g + g^{-1} A g \bar{A} - A \bar{A}) , \quad (2.3)$$

is invariant under the vector gauge transformations with parameters taking values in a subgroup H of G . Parametrising A and \bar{A} in terms of h and \tilde{h} from H , $A = h \partial h^{-1}$, $\bar{A} = \tilde{h} \bar{\partial} \tilde{h}^{-1}$ one can represent (2.3) as the difference of the two manifestly gauge-invariant terms: the ungauged WZW actions corresponding to the group G and the subgroup H ,

$$I(g, A) = kI_0(h^{-1} g \tilde{h}) - kI_0(h^{-1} \tilde{h}) . \quad (2.4)$$

This representation implies that the gauged WZW model corresponds to a conformal theory (coset CFT [58,59]). Fixing a gauge on g and changing the variables to $g' = h^{-1}g\tilde{h}$, $h' = h^{-1}\tilde{h}$ we get a σ -model on the group space $G \times H$ which is conformal to all orders in a particular ‘leading-order’ scheme. That means that the 1-loop group space solution remains exact solution in that scheme. Replacing (2.4) with the ‘quantum’ action with renormalised levels $k \rightarrow k + \frac{1}{2}c_G$ and $k \rightarrow k + \frac{1}{2}c_H$ does not change this conclusion. This replacement corresponds to starting with the theory formulated in the ‘CFT’ scheme in which, e.g., the exact central charge of the WZW model is reproduced by the first non-trivial correction [60,61] and the metric $(k + \frac{1}{2}c_G)G_{\mu\nu}$ is the one that appears in the CFT Hamiltonian L_0 considered as a Klein-Gordon operator.

2.1. ‘CFT’ and ‘leading-order’ schemes

To obtain the corresponding σ -model in the ‘reduced’ G/H configuration space (with coordinates being parameters of gauge-fixed g) one needs to integrate out A, \bar{A} (or, more precisely, the WZW fields h and \tilde{h}). This is a non-trivial step and the form of the result depends on a choice of a scheme in which the original ‘extended’ (g, h, \tilde{h}) WZW theory is formulated.

Suppose first the latter is taken in the leading-order scheme with the action (2.4). Then the result of integrating out A, \bar{A} and fixing a gauge takes the form of the σ -model (2.1) where the σ -model metric and dilaton are then given by (see [21] and refs. there)

$$G'_{\mu\nu} = G_{\mu\nu} - 2\alpha' \partial_\mu \phi \partial_\nu \phi, \quad \phi = \phi_0 - \frac{1}{2} \ln \det F. \quad (2.5)$$

$G_{\mu\nu}$ is the metric obtained by solving for A, \bar{A} at the classical level and ϕ_0 is the original constant dilaton. Since the α' -term in the metric can be eliminated by a field redefinition we conclude that there exists a *leading-order scheme* in which the leading-order gauged WZW σ -model background (G, B, ϕ) remains an exact solution. The leading-order scheme for the ungauged WZW σ -model is thus related to the leading-order scheme for the gauged WZW σ -model by an extra $2\alpha' \partial_\mu \phi \partial_\nu \phi$ redefinition of the metric. This provides a general explanation for the observations in [60,61] about the existence of a leading-order scheme for particular $D = 2, 3$ gauged WZW models.

If instead we start with the (g, h, \tilde{h}) WZW theory in the CFT scheme, i.e with the action

$$I(g, A) = (k + \frac{1}{2}c_G) [I_0(h^{-1}g\tilde{h}) - \frac{k + \frac{1}{2}c_H}{k + \frac{1}{2}c_G} I_0(h^{-1}\tilde{h})], \quad (2.6)$$

then the resulting σ -model couplings will explicitly depend on $1/k \sim \alpha'$ (and will agree with the coset CFT operator approach results [31,62,63,64]). While in the WZW model the transformation from the CFT to the leading order scheme is just a simple rescaling of couplings, this transformation becomes non-trivial at the level of gauged WZW σ -model. It is the ‘reduction’ of the configuration space resulting from integration over the gauge fields A, \bar{A} that is responsible for a complicated form of the transformation law between the ‘CFT’ and ‘leading-order’ schemes in the gauged WZW σ -models (in particular, this transformation involves dilaton terms of all orders in $1/k$, see below).⁵

⁵ An exception is provided by some σ -models obtained by nilpotent gauging: here the second term in (2.6) is absent by construction [34]. The background fields do not receive non-trivial $1/k$

The basic example is the $SL(2, R)/R$ gauged WZW model (or $D = 2$ black hole) [30,65,66]. The (euclidean) background in the leading-order scheme

$$ds^2 = dx^2 + \tanh^2 bx \, d\theta^2, \quad \phi = \phi_0 - \ln \cosh bx, \quad (2.7)$$

is related to the background in the CFT scheme [31]

$$ds^2 = dx^2 + \frac{\tanh^2 bx}{1 - p \tanh^2 bx} d\theta^2, \quad (2.8)$$

$$\phi = \phi_0 - \ln \cosh bx - \frac{1}{4} \ln(1 - p \tanh^2 bx), \quad (2.9)$$

$$p \equiv \frac{2}{k}, \quad \alpha' b^2 = \frac{1}{k-2}, \quad D - 26 + 6\alpha' b^2 = 0, \quad D = 2, \quad (2.10)$$

by the following local covariant redefinition [60]

$$G_{\mu\nu}^{(lead)} = G_{\mu\nu} - \frac{2\alpha'}{1 + \frac{1}{2}\alpha' R} [\partial_\mu \phi \partial_\nu \phi - (\partial\phi)^2 G_{\mu\nu}], \quad (2.11)$$

$$\phi^{(lead)} = \phi - \frac{1}{4} \ln(1 + \frac{1}{2}\alpha' R).$$

It should be emphasized that the two backgrounds related by (2.11) describe the same string geometry since the probe of the geometry is the tachyon field and the tachyon equation remains the same differential equation implied by coset CFT (even though it looks different being expressed in terms of different G and ϕ).

Similar remarks apply to a particular $D = 3$ solution – the ‘charged black string’ corresponding to $[SL(2, R) \times R]/R$ gauged WZW model [67]. Its form in the leading-order scheme can be obtained by making a duality rotation of the neutral black string, i.e. the direct product of the $D = 2$ black hole and a free scalar theory (see [61,68]).

2.2. Some $D = 4$ solutions

In general, the backgrounds obtained from the gauged WZW models have very few (at most, abelian) symmetries and thus cannot directly describe non-trivial $SO(3)$ -symmetric backgrounds which are of interest in connection with $D = 4$ cosmology and black holes. Let us mention the explicitly known $D = 4$ solutions with $(-, +, +, +)$ signature. A class of axisymmetric black-hole-type and anisotropic cosmological backgrounds is found by starting with $[G_1 \times G_2]/[H_1 \times H_2]$ gauged WZW model with various combinations of $G_i = SL(2, R)$ or $SU(2)$ and $H_i = R$ or $U(1)$ [33,69,70,71]. The leading-order form of these backgrounds can be obtained by applying the $O(2, 2)$ duality transformation to the direct product of the Euclidean and Minkowski $D = 2$ black holes or their analytic continuations. The stationary black-hole-type background is [33]

$$ds^2 = -\frac{g_1(y)}{g_1(y)g_2(z) - q^2} dt^2 + \frac{g_2(z)}{g_1(y)g_2(z) - q^2} dx^2 + dy^2 + dz^2, \quad (2.12)$$

corrections even in the CFT scheme, i.e. the relation between the leading-order and CFT schemes is equivalent to the one for the ungauged WZW model. The same is true for the $D = 3$ F -model or the extremal limit of the $SL(2, R) \times R/R$ coset.

$$B_{tx} = \frac{q}{g_1(y)g_2(z) - q^2}, \quad \phi = \phi_0 - \frac{1}{2}\ln(\sinh^2 y \sinh^2 z [g_1(y)g_2(z) - q^2]),$$

where $g_1 = \coth^2 y$, $g_2 = \coth^2 z$. The cosmological background [70] can be obtained by the analytic continuation and renaming the coordinates

$$ds^2 = -dt^2 + dx^2 + \frac{g_1(t)}{g_1(t)g_2(x) + q^2} dy^2 + \frac{g_2(x)}{g_1(t)g_2(x) + q^2} dz^2, \quad (2.13)$$

$$B_{yz} = \frac{q}{g_1(x)g_2(t) + q^2}, \quad \phi = \phi_0 - \frac{1}{2}\ln(\cos^2 t \cosh^2 x [g_1(t)g_2(x) + q^2]),$$

where $g_1 = \tan^2 t$, $g_2 = \tanh^2 x$. A non-trivial ('non-direct-product') $D = 4$ background without isometries is obtained from $SO(2, 3)/SO(1, 3)$ gauged WZW model [62].

Two other classes of $D = 4$ Minkowski signature 'coset' solutions will be discussed in more detail in the following sections. The first includes plane-wave-type backgrounds which are obtained from non-semisimple versions of $R \times SU(2)$ and the above $[SL(2, R) \times SL(2, R)]/[R \times R]$, etc., 'product' cosets [37,38,39,40,41,42,43]. The second contains F -models (backgrounds with 2 null Killing vectors) which follow from nilpotent gauging of rank 2 maximally non-compact groups [34].

Some other exact $D = 4$ solutions which are not described by gauged WZW models as a whole but incorporate $D = 2$ euclidean black hole background as a two-dimensional part will be mentioned below in Sections 4.3, 5.2, 8.

3. σ -models with conserved chiral null currents: general remarks

It appears that gauged WZW models are not sufficient in order to describe physically interesting $D \geq 4$ string solutions, e.g., black holes with asymptotically flat regions (cf. [72]) and non-trivial isotropic cosmological solutions (cf. [73,74]). We are thus to study other types of conformal σ -models which have physical signature and dimension. One possible direction of investigation (which remains largely unexplored) is to look for conformal σ -models [75,76,77,78] related to generalisations of coset models which solve of the affine-Virasoro master equation [79,78]. Here we shall discuss other classes of σ -models which are conformal because of their special *Minkowski* signature structure.

Every Killing vector generating a symmetry of spacetime fields gives rise to a conserved current on the string world sheet. If the antisymmetric tensor field is related to the spacetime metric in a certain way, these currents are chiral. The existence of such chiral currents turns out to simplify the search for exact conformal models. One example is the WZW model which describes string propagation on a group manifold where all the associated currents are chiral (since the gauged WZW models can be represented in terms of the difference between two WZW models for a group and a subgroup, a similar statement applies there). Another example is provided by the F -models discussed in [34,21,20] which have two null Killing vectors and two associated chiral currents. Also, the known plane wave solutions and their generalizations [80,24,25,26] are characterized by the existence of a covariantly constant null Killing vector.

F -models and generalized plane waves are both special cases of a larger class of exact solutions – 'chiral null models' [20] which have a null Killing vector and an associated conserved chiral current. The presence of a null chiral current is a consequence of an infinite-dimensional affine symmetry of the σ -model action. This symmetry is present if spacetime fields have certain special properties. The generalized connection with torsion equal to the antisymmetric field strength plays an important role since it is the one that

appears in the classical string equations of motion. This connection turns out to have reduced holonomy. A ‘balance’ between the metric and the antisymmetric tensor resulting in chirality of the action is the crucial property of the chiral null models which is in the core of their exact conformal invariance.

In addition to the plane-wave type solutions (Section 4) as well as the F -models [21] which contain the fundamental string solution [35] as a special case (Section 5), the class of chiral null models include generalizations of these solutions, e.g., the travelling waves along the fundamental string [81] (Section 6). Although the bosonic string does not have fundamental gauge fields, effective gauge fields can arise from dimensional reduction. In particular, charged fundamental string solutions [82,83] and four dimensional extreme electrically charged black holes [84,85,86] can also be obtained from the dimensional reduction of a chiral null model and hence are exact [36]. Similarly, the generalizations of the extremal black holes which include NUT charge and rotation [87,88,89] are also exact [20] (Section 7). Finally, the chiral null models and their generalisations describe also (electro)magnetic flux tube backgrounds [20,51,22] (Section 8).

If one considers only the leading order string equations, many of these solutions arise as an extremal limit of a family of solutions with a regular event horizon. It is clear, of course, that not all of the solutions of the leading order equations can be obtained from chiral null models. The chiral coupling leads to a no-force condition on the solutions, i.e. to the possibility of linear superposition. This happens only for a certain charge to mass ratio which typically characterizes extreme black holes or black strings. The balance between the spacetime metric and the antisymmetric tensor field necessary for the existence of a chiral current results upon dimensional reduction in a relation between the charge and the mass. Furthermore, one obtains only four dimensional black-hole type solutions with *electric* charge; extreme magnetically charged black holes do not appear to be described by chiral null models.

In general, the non-extremal solutions are not of the chiral null form and receive α' -corrections in all renormalization schemes. Finding the exact analogs of these solutions (which include as a special case the Schwarzschild metric) remains an outstanding open problem.

Below we shall first discuss the plane wave and F -model subclasses of the class of chiral null models and then discuss some special cases and generalisations.

4. Solutions with covariantly constant null Killing vector

The string action in a generic background with a covariantly constant null Killing vector can be represented in the following form, cf. (2.1) ($i, j = 1, \dots, N$)

$$L = \partial u \bar{\partial} v + K(u, x) \partial u \bar{\partial} u + 2A_i(u, x) \partial u \bar{\partial} x^i + 2\bar{A}_i(u, x) \bar{\partial} u \partial x^i \\ + (G_{ij} + B_{ij})(u, x) \partial x^i \bar{\partial} x^j + \mathcal{R}\phi(u, x) . \quad (4.1)$$

K, A_i, \bar{A}_i can be eliminated (locally) by a coordinate and $B_{\mu\nu}$ -gauge transformation [80,29] so that the general form of the Lagrangian is

$$L = \partial u \bar{\partial} v + (G_{ij} + B_{ij})(u, x) \partial x^i \bar{\partial} x^j + \mathcal{R}\phi(u, x) , \quad (4.2)$$

with K, A_i, \bar{A}_i been now ‘hidden’ in a possible coordinate and gauge transformation. If the ‘transverse’ space is trivial (for fixed u) one may use (4.2) with ‘flat’ G_{ij}, B_{ij} taken in the most general frame, or may choose a special frame where $G_{ij} = \delta_{ij}, B_{ij} = 0$ and then go back to (4.1) (which may be preferable for a global coordinate choice).

There are two possibilities to satisfy the conditions of conformal invariance. The first (which will be discussed in the following subsections) is realised in the case when the ‘transverse’ theory is conformally invariant. The second is realised when the ‘transverse’ N -dimensional part of the model is not conformal but one may still find a conformal model by adding the term linear in v to the dilaton field [90]. Then the conformal invariance conditions are satisfied provided the ‘transverse’ couplings G, B, ϕ depend on u according to the standard RG equations with the β -functions of the ‘transverse’ model, $\partial G_{ij}/\partial u = \beta_{ij}^G$, etc. Since in general these β -functions are not known in a closed form one is unable to determine the explicit all-order form of the solution. An exception is provided by the theories with transverse part represented by $(2, 2)$ supersymmetric Einstein-Kähler σ -models [90] (see also [91]) when the exact β -function is given by the one-loop term. The simplest non-trivial example corresponds to the case when the transverse theory is the $O(3)$ supersymmetric σ -model. The resulting metric is that of $D = N + 2 = 4$ dimensional space with the transverse part being proportional to the metric on S^2 ,

$$ds^2 = dudv + u(d\theta^2 + \sin^2\theta d\varphi^2) . \quad (4.3)$$

To get a solution of the conformal invariance conditions one should add the following dilaton

$$\phi(v, u) = \phi_0 + v + \frac{1}{4}\ln u . \quad (4.4)$$

The spacetime (4.3) is conformal to the direct product of two-dimensional Minkowski space and two-sphere. Though the exact metric has apparent curvature singularity at $u = 0$ the corresponding CFT may be non-singular. Note also that the string coupling $e^\phi = Au^{1/4}e^v$ goes to zero in the strong coupling region $u \rightarrow 0$ of the transverse sigma model, i.e. is small near the singularity $u = 0$. Similar remarks apply to some other exact solutions discussed below, e.g., to the fundamental string one.

4.1. Plane waves

The simplest special case of (4.2) is that of the ‘plane-wave’ backgrounds for which G, B, ϕ in (4.2) do not depend on x (so that, in particular, the transverse metric is flat). The conformal invariance condition then reduces to one equation

$$-\frac{1}{2}G^{ij}\ddot{G}_{ij} + \frac{1}{4}G^{ij}G^{mn}(\dot{G}_{im}\dot{G}_{jn} - \dot{B}_{im}\dot{B}_{jn}) + 2\ddot{\phi} = 0 . \quad (4.5)$$

$G_{ij}(u), B_{ij}(u)$ and $\phi(u)$ satisfying (4.5) represent exact solutions of string theory [92] (which transform into each other under the full duality symmetry group $O(N, N; R)$).

A subclass of such models admits a coset CFT interpretation in terms of gauged WZW models for non-semisimple groups [37,38,39,40,93,41,42,43]. The latter can be obtained from standard semisimple gauged WZW models by taking special singular limits. The singular procedure involves a coordinate transformation and a rescaling of α' and is carried out directly at the level of the string action [40,43].⁶ Using the singular limit one is able to obtain the plane-wave background fields and formal CFT operator algebra relations but to construct the full ‘non-semisimple’ coset CFT from a ‘semisimple’ one may be a more subtle problem⁷ so that one may need to start directly from the ‘non-semisimple’ coset CFT in order to give a CFT description to this subclass of plane-wave solutions (see [39]).

⁶ It is similar to the transformation in [94] which maps any space-time into some plane wave.

⁷ This singular limit changes the boundary conditions or topology (mapping, e.g., a compact coordinate into a non-compact one), so it is not a priori clear that the correspondence between the two conformal models holds also at the full CFT level, i.e. at the level of states and correlation functions.

In particular, a set of $D = 4$ plane wave solutions that can be obtained in this way from the gauged $[SU(2) \times SL(2, R)]/[U(1) \times R]$ WZW models is given by [43] (cf.(2.13))

$$ds^2 = dudv + \frac{g_1(u')}{g_1(u')g_2(u) + q^2} dx_1^2 + \frac{g_2(u)}{g_1(u')g_2(u) + q^2} dx_2^2 , \quad (4.6)$$

$$B_{12} = \frac{q}{g_1(u')g_2(u) + q^2} , \quad \phi = \phi_0 - \frac{1}{2} \ln(f_1^2(u')f_2^2(u)[g_1(u')g_2(u) + q^2]) ,$$

where $u' = au + d$ ($a, d = \text{const}$) and the functions g_i, f_i can take any pair of the following values

$$\begin{aligned} g(u) &= 1 , \quad u^2 , \quad \tanh^2 u , \quad \tan^2 u , \quad u^{-2} , \quad \coth^2 u , \quad \cot^2 u , \\ f(u) &= 1 , \quad 1 , \quad \cosh u , \quad \cos u , \quad u , \quad \sinh u , \quad \sin u . \end{aligned} \quad (4.7)$$

A particular case is $g_1 = 1, g_2 = u^2$ (this background is dual to flat space). Another special case is the E_2^c WZW model of [37]⁸

$$L = \partial u \bar{\partial} v + \partial x_1 \bar{\partial} x_1 + \partial x_2 \bar{\partial} x_2 + 2 \cos u \partial x_1 \bar{\partial} x_2 , \quad (4.8)$$

which can be obtained by a singular limit from the WZW action for $SU(2) \times R$.

The special case of (4.2) with $G_{ij} = G_{ij}(u)$, $B_{ij} = 0$, $\phi = \phi(u)$ or, equivalently, (4.1) with $G_{ij} = \delta_{ij}$, $A_i = \bar{A}_i = B_{ij} = 0$, $\phi = \phi(u)$ is

$$L = \partial u \bar{\partial} v + K(u, x) \partial u \bar{\partial} u + \partial x^i \bar{\partial} x_i + \mathcal{R} \phi(u) . \quad (4.9)$$

The fact that this model has a covariantly constant null vector can be used to give a simple geometrical argument [25,26] that such leading order solutions are exact. The curvature contains two powers of the constant null vector l , and derivatives of ϕ are also proportional to l . One can thus show that all higher order terms in the equations of motion vanish identically. The one-loop conformal invariance condition

$$-\frac{1}{2} \partial^i \partial_i K + 2 \partial_u^2 \phi = 0 , \quad (4.10)$$

is then the exact one and is solved, e.g., by the standard plane-wave ansatz [23,25,26] $K = w_{ij}(u)x^i x^j$. A different rotationally symmetric solution exists for $\phi = \text{const}$ ($r^2 \equiv x_i x^i > 0$)

$$K = 1 + \frac{M}{r^{D-4}} , \quad D > 4 ; \quad K = 1 - \mu \ln \frac{r}{r_0} , \quad D = 4 . \quad (4.11)$$

This background [84] is dual [95] to the fundamental string solution and describes a string boosted to the speed of light.⁹

⁸ E_2^c is a $D = 4$ non-semisimple algebra which is a central extension of the Euclidean algebra in two dimensions admitting a non-degenerate invariant bilinear form.

⁹ Note that this solution needs a source to support it at the origin.

4.2. Generalised plane waves

If one sets $G_{ij} = \delta_{ij}$, $B_{ij} = 0$, $\phi = \phi(u)$ in (4.1) but keeps A_i, \bar{A}_i non-vanishing the conditions of conformal invariance take the form [29,20] ($\mathcal{F}_{ij} \equiv \partial_i A_j - \partial_j A_i$, $\bar{\mathcal{F}} \equiv \partial_i \bar{A}_j - \partial_j \bar{A}_i$)

$$\partial^j \mathcal{F}_{ij} = 0, \quad \partial^j \bar{\mathcal{F}}_{ij} = 0, \quad (4.12)$$

$$-\frac{1}{2} \partial^i \partial_i K + \mathcal{F}^{ij} \bar{\mathcal{F}}_{ij} + \partial^i \partial_u (A_i + \bar{A}_i) + 2 \partial_u^2 \phi + \sum_{s=1}^{\infty} c_s \alpha'^{2s} O(\partial^s \mathcal{F} \partial^s \bar{\mathcal{F}}) = 0. \quad (4.13)$$

These equations admit straightforward generalisation to the case of $\phi = \phi(u) + b_i x^i$. The higher order terms do not vanish in general so it is not true that any model with a covariantly constant Killing vector does not receive α' corrections. However, α'^n -terms in (4.13) have particular structure and thus vanish, e.g., for $\bar{A}_i = 0$. The special property of the ‘chiral’ model with $\bar{A}_i = 0$ or $A_i = 0$ (i.e. with $G_{ui} = \pm B_{ui}$) resulting in cancellation of the vector-dependent contributions to the β_{uu} -function corresponding to the renormalisation of K was noted at the one-loop level in [29] and extended to the two-loop level in [96].¹⁰ It was further shown [98] that such backgrounds are (‘half’) supersymmetric when embedded in $D = 10$ supergravity theory and it was suggested that these ‘supersymmetric string waves’ remain exact heterotic string solutions to all orders in α' when supplemented with some gauge field background. It was shown in [20] that ‘chiral’ plane wave backgrounds with $\bar{A}_i = 0$ (or $A_i = 0$)

$$L = \partial u \bar{\partial} v + K(u, x) \partial u \bar{\partial} u + 2 A_i(u, x) \partial u \bar{\partial} x^i + \partial x^i \bar{\partial} x^i + \mathcal{R} \phi(u), \quad (4.14)$$

are exact solutions of bosonic string theory provided K, A_i, ϕ satisfy the one-loop conformal invariance conditions,

$$-\frac{1}{2} \partial^2 K + \partial^i \partial_u A_i + 2 \partial_u^2 \phi = 0, \quad \partial^j \mathcal{F}_{ij} = 0. \quad (4.15)$$

Furthermore, due to the chiral structure of the model (4.14), they can be promoted to the exact superstring and heterotic string solutions even without an extra gauge field background [20].

The models (4.14) with trivial K and A_i with a constant field strength,

$$K = 0, \quad A_i = -\frac{1}{2} \mathcal{F}_{ij} x^j, \quad \mathcal{F}_{ij} = \text{const}, \quad (4.16)$$

can be interpreted [20] as boosted products of group spaces, or, equivalently, as spaces corresponding to WZW models for non-semisimple groups¹¹, with the $D = 4$ model of [37]

¹⁰ As explained in [29], the exact solutions of the string equations with $G_{ij} = \delta_{ij}$, $A_i = -\frac{1}{2} \mathcal{F}_{ij}(u) x^j$, $K = K_0 + k_{ij} x^i x^j$, considered in [24,25,26] are the same as the solutions with $g_{ij} = g_{ij}(u)$ in [27,97,96]. Equivalent representations for $B_{\mu\nu}$ are $B_{iu} = -\frac{1}{2} \dot{b}_{ij} x^j$, $B_{ij} = 0$ and $B_{iu} = 0$, $B_{ij} = b_{ij}(u)$.

¹¹ In even dimensional cases these are Heisenberg groups with the algebra containing generators (e_0, e'_0, e_i, e'_i) with non-trivial commutation relations (e_0 belongs to the center): $[e'_0, e_i] = -e_i$, $[e'_0, e'_i] = e'_i$, $[e_i, e'_i] = \delta_{ij} e_0$ (see Kehagias and Meessen, ref. [41]).

being the simplest ($D = 4$, $N = 2$) example ($A_i = -\frac{1}{2}\beta\epsilon_{ij}x^j$; β can be set equal to 1 by rescaling u and v). In particular, the corresponding connection with torsion is flat. These σ -models (or corresponding non-semisimple WZW models) can be obtained by singular boosts and rescalings of levels from semisimple WZW models based on direct products of $SL(2, R)_{-k}$, $SU(2)_k$ and R factors. It is interesting to note also that all these models (4.14), (4.16), can be formally related by $O(d, d; R)$ duality to the flat space model [20] in the same way as this was shown [39, 34] for the $D = 4$ model of [37] (see also Section 9.1).

The higher-loop corrections in (4.13) vanish also in another special case when both A_i and \bar{A}_i have field strengths¹² constant in x^i direction (in general, the field strengths and ϕ may still depend on u but for simplicity we shall assume that they are constant)

$$A_i = -\frac{1}{2}\mathcal{F}_{ij}x^j, \quad \bar{A}_i = -\frac{1}{2}\bar{\mathcal{F}}_{ij}x^j. \quad (4.17)$$

Then (4.13) takes the form

$$-\frac{1}{2}\partial^i\partial_i K + \mathcal{F}^{ij}\bar{\mathcal{F}}_{ij} = 0, \quad (4.18)$$

$$K = K_0(x) + mx^i x_i, \quad m \equiv \frac{1}{2}\mathcal{F}^{ij}\bar{\mathcal{F}}_{ij}, \quad \partial^i\partial_i K_0 = 0. \quad (4.19)$$

One can choose, e.g., $K_0 = k_0 = \text{const}$ or $K_0 = k_0 + M/r^{D-4}$ as in (4.11). This model

$$L = \partial u \bar{\partial} v + K(x) \partial u \bar{\partial} u + 2A_i(x) \partial u \bar{\partial} x^i + 2\bar{A}_i(x) \bar{\partial} u \partial x^i + \mathcal{R}\phi_0, \quad (4.20)$$

represents a simple and interesting conformal theory which (for $K_0 = \text{const}$) can be solved explicitly [51].¹³ This model and a similar one obtained by applying the duality transformation in u belong to a generalisation of chiral null models discussed below in Section 8.

4.3. K -models with curved transverse space

Another special case of (4.1) is found when K, G_{ij}, B_{ij}, ϕ do not depend on u but G, B are, in general, nontrivial functions of x^i . This is a generalisation of the plane wave solutions called the ‘ K -model’ in [21]¹⁴

$$L_K = \partial u \bar{\partial} v + K(x) \partial u \bar{\partial} u + (G_{ij} + B_{ij})(x) \partial x^i \bar{\partial} x^j + \mathcal{R}\phi(x). \quad (4.21)$$

Suppose that the transverse space is known to be an exact string solution in some scheme. The exact form of the conformal invariance equation for K turns out to be the following [21]

$$-\omega K + \partial^i \phi \partial_i K = 0, \quad \omega = \frac{1}{2}\nabla^2 + O(\alpha'), \quad (4.22)$$

¹² More generally, the higher loop corrections to (4.13) vanish when at least one of the two field strengths is constant. To get a solution the non-constant field strength should still satisfy the Maxwell equation (4.12); for example, in $D = 3$ it may correspond to a monopole background (cf. (7.18), (7.19)).

¹³ One particular case of it corresponds to the $D = 4$ E_2^c WZW model [37], namely, $K = -x^i x_i$, $A_i = -\bar{A}_i = -\frac{1}{2}\epsilon_{ij}x^j$, $\phi = \text{const}$, which is obviously a solution of (4.18). This representation is related by a u -dependent coordinate transformation of x^i to another one mentioned above: $K = 0$, $A_i = -\frac{1}{2}\epsilon_{ij}x^j$, $\bar{A}_i = 0$.

¹⁴ K -term cannot be eliminated as in (4.2) since G, B are assumed to be u -independent.

where ω is the scalar anomalous dimension operator which in general contains (G_{ij}, B_{ij}) -dependent corrections to all orders in α' (only several leading α'^n -terms in it are known explicitly, see, e.g., [60] and refs. there). The simplest possibility is that of the flat transverse space with linear dilaton ($\phi = \phi_0 + b_i x^i$) which is an obvious generalisation of the plane wave case. It is possible to obtain more interesting exact solutions when the CFT behind the ‘transverse’ space solution (G_{ij}, B_{ij}, ϕ) is nontrivial but is known explicitly [21]: in that case the structure of the ‘tachyonic’ operator ω is determined by the zero mode part of the CFT Hamiltonian $H = L_0 + \tilde{L}_0$. Fixing a particular scheme (e.g., the ‘CFT’ one where H has the standard Klein-Gordon form with the dilaton term) one is able, in principle, to establish the exact form of the background fields (G_{ij}, B_{ij}, ϕ) and K . This produces ‘hybrids’ of gauged WZW and plane wave solutions.

To obtain a *four* dimensional hybrid K -model solution [21] one must start with a two dimensional conformal σ -model. Essentially the only (up to analytic continuation) non-trivial possibility is the $SL(2, R)/U(1)$ gauged WZW model which describes the two dimensional euclidean black hole. The exact background fields of the $SL(2, R)/U(1)$ model in the CFT scheme were given in (2.8), (2.9). In the CFT scheme the tachyonic equation has the standard uncorrected form, so that the function $K(x)$ must satisfy

$$-\frac{1}{2}\nabla^2 K + \partial^i \phi \partial_i K = -\frac{1}{2\sqrt{G}e^{-2\phi}} \partial_i (\sqrt{G}e^{-2\phi} G^{ij} \partial_j) K = 0 . \quad (4.23)$$

A particular solution of (4.23) with K depending only on x and not on θ is

$$K = a + m \ln \tanh bx . \quad (4.24)$$

The constants a, m can be absorbed into a redefinition of u and v , so that the full exact $D = 4$ metric is [21]

$$ds^2 = dudv + \ln \tanh bx \, du^2 + dx^2 + \frac{\tanh^2 bx}{1 - p \tanh^2 bx} d\theta^2 , \quad (4.25)$$

while the dilaton is unchanged. This metric is asymptotically flat, being (at $x \rightarrow \infty$) a product of $D = 2$ Minkowski space with a cylinder.

The solution for K is the *same* in the ‘leading-order’ scheme where the metric and dilaton do not receive α' corrections. The reason is that the differential operator in the tachyon equation remains the same, it is only its expression in terms of the new G, ϕ that changes. Thus, in the ‘leading-order’ scheme we get the following exact $D = 4$ solution

$$ds^2 = dudv + \ln \tanh bx \, du^2 + dx^2 + \tanh^2 bx \, d\theta^2 , \quad \phi = \phi_0 - \ln \cosh bx . \quad (4.26)$$

In addition to the covariantly constant null vector $\partial/\partial v$, this solution has two isometries corresponding to the shifts of u and θ , i.e. one can consider two different types of duals. The u -dual background will be given in Section 5.

5. Solutions described by F -models

5.1. Simple F -model: fundamental string solution

The simplest F -model [34, 21]

$$L_F = F(x) \partial u \bar{\partial} v + \partial x^i \bar{\partial} x_i + \mathcal{R} \phi(x) , \quad (5.1)$$

describes a family of backgrounds with metric and antisymmetric tensor characterized by a single function $F(x)$ and dilaton $\phi(x)$ (depending only on the transverse coordinates x^i , $i = 1, \dots, N$)

$$ds^2 = F(x)dudv + dx_i dx^i, \quad B_{uv} = \frac{1}{2}F(x). \quad (5.2)$$

The leading order string equations then reduce to [34]

$$F^2(-\frac{1}{2}\partial^2 F^{-1} + b^i \partial_i F^{-1}) = 0, \quad \phi = \phi_0 + b_i x^i + \frac{1}{2}\ln F(x), \quad (5.3)$$

where b_i is a constant vector. At the points where F is nonvanishing we get

$$\partial^2 F^{-1} = 2b^i \partial_i F^{-1}. \quad (5.4)$$

Some of the solutions to (5.3) were shown to correspond to gauged WZW models with the gauged subgroup being nilpotent [34] (see Section 5.3 below). It was argued that like ungauged WZW models they should not receive non-trivial higher order corrections even in the CFT scheme (cf. Section 2.1). Though not all of the solutions to (5.3) can be obtained from gauged WZW models one can show [21] that indeed there exists a scheme in which all of them are exact and receive no α' corrections.¹⁵ Since the equation (5.4) for F^{-1} is linear, linear combinations of these solutions yield new exact solutions.

One of the most interesting solutions in this class is the ‘fundamental string’ (FS) one [35] which has $b_i = 0$ and ($r^2 = x_i x^i$, $D = 2 + N$)

$$F^{-1} = 1 + \frac{M}{r^{D-4}}, \quad D > 4; \quad F^{-1} = 1 - \mu \ln \frac{r}{r_0}, \quad D = 4. \quad (5.5)$$

This solution can be interpreted [100,35] as the field of a straight fundamental string located at $r = 0$. Note that in contrast to [35] where the leading-order equation which follows from the effective action (and thus has upper indices, $R^{\mu\nu} + \dots = \text{source}$) was solved, one does not need a source at the origin to satisfy the conformal invariance condition (5.3) ($R_{\mu\nu} + \dots = 0$ everywhere, including $r = 0$). To prove exactness of the fundamental string solution [21] it was important to write down the corresponding string σ -model which took the very simple F -model form (5.1), having two ‘null’ chiral currents.

5.2. General F -model

The key property of the K -model (4.21) is that it has a covariantly constant null vector $\partial/\partial v$. The main features of the F -model are that there are two null Killing vectors corresponding to translations of u and v , and that the coupling to u, v is chiral (i.e. $G_{uv} = B_{uv}$). This means that in addition to the three global Poincare transformations in the u, v plane the F -model is invariant under the infinite dimensional symmetry $u' = u + f(\tau - \sigma)$ and $v' = v + h(\tau + \sigma)$. Associated with this symmetry are two conserved world sheet chiral currents: $\bar{J}_u = F\bar{\partial}v$, $J_v = F\partial u$. These properties are preserved if the transverse x^i -space is curved, i.e. the general F -model is defined as [21]

$$L_F = F(x)\partial u \bar{\partial} v + (G_{ij} + B_{ij})(x)\partial x^i \bar{\partial} x^j + \mathcal{R}\phi(x). \quad (5.6)$$

¹⁵ The central observation was that according to [99] the determinant resulting from integration over u and v is local and (after a local change of a scheme) its only effect is to cancel the nontrivial dilaton term in (5.1).

The all-order conformal invariance conditions for F -model are satisfied [21] provided one is given a conformal ‘transverse’ theory (G, B, ϕ') and (assuming a special ‘leading-order’ scheme is used)

$$\phi = \phi' + \frac{1}{2} \ln F , \quad (5.7)$$

$$-\omega F^{-1} + \partial^i \phi' \partial_i F^{-1} = -\frac{1}{2} [\nabla^2 + O(\alpha')] F^{-1} + \partial^i \phi' \partial_i F^{-1} = 0 . \quad (5.8)$$

As in (4.22) ω is the scalar anomalous dimension operator of the transverse theory, depending, in general, on G_{ij} and B_{ij} . When (G, B, ϕ') correspond to a known (e.g., coset) CFT this equation can be put in an explicit form to all orders in α' .

The model (5.6) is related by leading order duality to K -model (4.21): the dual of (4.21) with respect to u is (5.6) with $F = K^{-1}$. In the leading-order scheme the leading-order duality is exact. In particular, (5.6) with $G_{ij} = \delta_{ij}$, $B_{ij} = 0$ is dual to the K -model (cf. (4.21))

$$L_K = \partial u \bar{\partial} v + F^{-1}(x) \partial u \bar{\partial} u + \partial x^i \bar{\partial} x_i + \mathcal{R}(\phi_0 + b_i x^i) , \quad (5.9)$$

which represents an exact solution if F solves (5.3).

Another exact $D = 4$ F -model solution is obtained by taking the transverse two dimensional theory to be non-trivial, i.e. represented by the $D = 2$ euclidean black hole background [21]. This solution can be obtained from (4.21) with K given by (4.24) by the u -duality transformation (cf. (4.26))

$$ds^2 = (a + m \ln \tanh bx)^{-1} du dv + dx^2 + \tanh^2 bx \, d\theta^2 , \quad (5.10)$$

$$B_{uv} = \frac{1}{2} (a + m \ln \tanh bx)^{-1} , \quad \phi = \phi_0 - \ln \cosh bx + \frac{1}{2} \ln F .$$

The background (5.10) can be viewed as a generalisation of the fundamental string (5.5) in four dimensions (with $F^{-1} = 1 - \mu \ln(r/r_0)$). While the latter has, in addition to the usual singularity at $r = 0$, another singularity outside the string at $r = r_0$, the solution (5.10) has the same singularity at the origin but is regular elsewhere and is asymptotically flat. The original fundamental string solution can be recovered by taking the limit $b \rightarrow 0$ which is consistent since the central charge condition is now imposed only at the level of the full $D = 4$ solution.

5.3. F -models obtained by nilpotent gauging of WZW models

The conclusion about the existence of a scheme where the F -model (5.1), (5.3) represents an exact string solution is consistent with the result of [34] that the particular model (5.1) with

$$F^{-1} = \sum_{i=1}^N \epsilon_i e^{\alpha_i \cdot x} , \quad \phi = \phi_0 + \rho \cdot x + \frac{1}{2} \ln F , \quad (5.11)$$

can be obtained from a G/H gauged WZW model. Here the constants ϵ_i take values 0 or ± 1 , α_i are simple roots of the algebra of a maximally non-compact real Lie group G of rank $N = D - 2$ and $\rho = \frac{1}{2} \sum_{s=1}^m \alpha_s$ is half of the sum of all positive roots. H is a nilpotent subgroup of G generated by $N - 1$ simple roots (this condition on H is needed to get a model with only one time-like direction). The flat transverse coordinates x^i correspond to the Cartan subalgebra generators.

This ‘null’ gauging [34] is based on the Gauss decomposition and thus directly applies only to the groups with the algebras that are the ‘maximally non-compact’ real forms of the

Lie algebras (real linear spans of the Cartan-Weyl basis), i.e. $sl(N+1, R)$, $so(N, N+1)$, etc. These WZW models can be considered as natural generalisations of the $SL(2, R)$ WZW model. For such groups there exists a real group-valued Gauss decomposition

$$g = \exp\left(\sum_{\Phi_+} u^\alpha E_\alpha\right) \exp\left(\sum_{i=1}^N x^i H_i\right) \exp\left(\sum_{\Phi_+} v^\alpha E_{-\alpha}\right).$$

Φ_+ is the set of positive roots of a complex algebra with the Cartan-Weyl basis consisting of the step operators E_α , $E_{-\alpha}$, $\alpha \in \Phi_+$ and $N(=\text{rank}G)$ Cartan subalgebra generators H_i .

The *four* dimensional ($D = 2 + N = 4$) models are obtained for each of the rank 2 maximally non-compact groups: $SL(3, R)$, $SO(2, 3) = Sp(4, R)$, $SO(2, 2) = SL(2, R) \oplus SL(2, R)$ and G_2 . In the rank 2 case the corresponding background is parametrised by a 2×2 Cartan matrix or by two simple roots with components α_{1i} and α_{2i} and one parameter $\epsilon = \epsilon_2/\epsilon_1$ with values ± 1 ,

$$L = (e^{\alpha_1 \cdot x} + \epsilon e^{\alpha_2 \cdot x})^{-1} \partial u \bar{\partial} v + \partial x_1 \bar{\partial} x_1 + \partial x_2 \bar{\partial} x_2 \quad (5.12)$$

$$+ \alpha' \mathcal{R}[\phi_0 + \frac{1}{2}(\alpha_1 \cdot x + \alpha_2 \cdot x) - \frac{1}{2} \ln(e^{\alpha_1 \cdot x} + \epsilon e^{\alpha_2 \cdot x})].$$

For example, in the case of $SL(3, R)$ $\alpha_1 = (\sqrt{2}, 0)$, $\alpha_2 = (-1/\sqrt{2}, 3/\sqrt{2})$. In addition to the Poincare symmetry in the u, v plane and affine transformations of u and v this model is also invariant under a correlated constant shift of x^i and u (or v). For $\epsilon = 1$ the curvature is non-singular in the coordinate patch used in (5.1), but there is also a horizon so one should first consider the geodesic completion. It is likely that there is still no curvature singularity, as in the case of the $SL(2, R)$ group space (in any case, the corresponding CFT seems to be non-singular).

The presence of the two chiral currents implies [34] that the classical equations following from (5.1), (5.11) which describe the string propagation in these backgrounds reduce to the Toda equation for x^i (cf. [101])

$$\partial \bar{\partial} x_i + \frac{1}{2} \chi \partial_i F^{-1} = 0, \quad F \bar{\partial} v = \nu(\bar{z}), \quad F \partial u = \mu(z), \quad \chi \equiv \nu(\bar{z}) \mu(z), \quad (5.13)$$

where χ can be made constant by the (conformal) coordinate transformation. Then the solutions (including the solutions of the constraints) can be expressed in terms of the Toda model solutions.

Let us mention also that there exists another example of F -model which corresponds to a gauged WZW model: the general F -model in 3 dimensions. Solving (5.4) in $D = 3$ one finds that $F^{-1} = a + m e^{bx}$. As explained in [21, 20], this model can be obtained from a special $SL(2, R) \times R/R$ gauged WZW model and, at the same time, is the extremal limit of the charged black string solution [67]. The case of $a = 0$, i.e. (5.1) with $F = e^{-bx}$, corresponds to the ungauged $SL(2, R)$ WZW model (with the action written in the Gauss decomposition parametrisation).

6. Chiral null models

Chiral null models [20] is a class of σ -models which generalize both plane wave models and F -models and have one conserved chiral null current (cf. (4.14), (5.1))

$$L = F(x) \partial u \bar{\partial} v + \tilde{K}(u, x) \partial u \bar{\partial} u + 2 \tilde{A}_i(u, x) \partial u \bar{\partial} x^i + \partial x^i \bar{\partial} x^i + \mathcal{R} \phi(u, x), \quad (6.1)$$

or, with $\tilde{K} \equiv FK$, $\tilde{A}_i \equiv FA_i$,

$$L = F(x)\partial u[\bar{\partial}v + K(u, x)\bar{\partial}u + 2A_i(u, x)\bar{\partial}x^i] + \partial x^i\bar{\partial}x^i + \mathcal{R}\phi(u, x) . \quad (6.2)$$

For simplicity we assumed that the transverse space is flat but a generalisation to the case of any conformal transverse theory is straightforward. Note that F does not depend on u since otherwise the model is not conformal. Like the F -term, the vector coupling has a special chiral structure: the G_{ui} and B_{ui} components of the metric and the antisymmetric tensor are equal to each other. The affine symmetry $v' = v + h(\tau + \sigma)$ of the action implies the existence of a conserved chiral null current. In particular, the metric has the null Killing vector and the corresponding string background has special holonomy properties. The action is also invariant under the subgroup of coordinate transformations combined with a ‘gauge transformation’ of the coupling functions

$$v' = v - 2\eta(x, u) , \quad K' = K + 2\partial_u\eta , \quad A'_i = A_i + \partial_i\eta . \quad (6.3)$$

Using this freedom one may choose a gauge in which $K = 0$. This is not possible in the special case (which we shall mostly consider below) when K , A_i and ϕ do not depend on u , i.e. when $\partial/\partial u$ is a Killing vector and K cannot be set to zero without loss of generality. In the latter case the chiral null model is ‘self-dual’: performing a leading-order duality transformation along any non-null direction in the (u, v) -plane (e.g. setting $v = \hat{v} + au$, and dualizing with respect to u) yields a σ -model of exactly the same form with F , K , A_i and ϕ replaced by

$$F' = (K + a)^{-1} , \quad K' = F^{-1} , \quad A'_i = A_i , \quad \phi' = \phi - \frac{1}{2}\ln[F(K + a)] . \quad (6.4)$$

As in the case of the F -model, there exists a scheme in which the conditions on the functions F , K , A_i and ϕ under which (6.2) is conformal to all orders in α' turn out to be equivalent to the leading-order equations [20]

$$F^2(-\frac{1}{2}\partial^2 F^{-1} + b^i\partial_i F^{-1}) = 0 , \quad F(-\frac{1}{2}\partial_i \mathcal{F}^{ij} + b_i \mathcal{F}^{ij}) = 0 , \quad (6.5)$$

$$F(-\frac{1}{2}\partial^2 K + b^i\partial_i K + \partial^i\partial_u A_i - 2b^i\partial_u A_i) + 2\partial_u^2 \phi = 0 , \quad (6.6)$$

$$\phi(u, x) = \phi(u) + b_i x^i + \frac{1}{2}\ln F(x) , \quad (6.7)$$

$$\mathcal{F}_{ij} \equiv \partial_i A_j - \partial_j A_i , \quad \partial^2 \equiv \partial^i \partial_i .$$

Eq. (6.7) implies that the central charge of the model is $c = D + 6b^i b_i$. When F, K, A_i, ϕ are independent of u and $b_i = 0$, these equations take simpler form (at the points where F is non-vanishing)

$$\partial^2 F^{-1} = 0 , \quad \partial^2 K = 0 , \quad \partial_i \mathcal{F}^{ij} = 0 , \quad \phi = \phi_0 + \frac{1}{2}\ln F(x) . \quad (6.8)$$

Since these equations are linear their solutions satisfy a solitonic no-force condition and can be superposed.

Special cases of the model (6.2) include chiral plane waves ($F = 1$) and generalisations of the fundamental string solution, e.g., describing travelling waves along fundamental string, $A_i = 0$, $F^{-1} = 1 + M/r^{D-4}$, $K = K(u, x)$ (some of these solutions were originally found as the leading-order solutions of the string effective equations in [81,82,83,102]). In

particular, in the u -independent spherically symmetric case $K = a + b/r^{D-4} = a' + b'F^{-1}$ so that (redefining u and v) the action (6.2) can be put into the simple form

$$L = F(x)\partial u\bar{\partial}v + n\partial u\partial u + \partial x_i\bar{\partial}x^i + \mathcal{R}\phi(x) , \quad n = 0, \pm 1 , \quad (6.9)$$

where the three values of n correspond to inequivalent models.¹⁶

Other related solutions will be discussed in detail in the following section.

7. $D = 4$ solutions with electromagnetic fields corresponding to $D > 4$ chiral null models

There exist several classes of exact $D = 5$ solutions which ‘reduce’ to interesting $D = 4$ backgrounds if the size of the compact 5-th dimension is very small. Every solution with an asymptotically flat four dimensional space-time must have a number of internal dimensions (to have the total central charge 26 or 10). Following Kaluza-Klein idea, we may assume that off-diagonal components of the higher dimensional fields give rise to extra four dimensional fields, in particular, gauge fields. We shall consider the closed bosonic string theory which has no fundamental gauge fields in the higher dimensional space but the gauge fields do appear upon dimensional reduction when the theory is ‘viewed’ from four dimensions. The ‘reduction’ does not mean that we drop the dependence on the internal direction: the conformal σ -models which describe these exact $D = 4$ solutions are still higher dimensional ones.

Below we shall first review the procedure of ‘dimensional reduction’ in string theory and then summarize the information about string black hole solutions which are exact to all orders in α' . All known exact black hole solutions turn out to be the extreme ones, with electric solutions being described by $D = 5$ chiral null models [36,20].

7.1. Kaluza-Klein ‘reduction’

The procedure of ‘dimensional reduction’ (e.g. from $D = 5$ to $D = 4$) in string theory is reinterpreting the $D = 5$ bosonic string σ -model action (with background fields not depending on x^5) as an action of a $D = 4$ string with an internal degree of freedom (represented by the Kaluza-Klein coordinate x^5) which describes the coupling to additional vector (and scalar) background fields. Explicitly, this corresponds to the following ‘Kaluza-Klein’ rearrangement of terms in the $D = 5$ σ -model action (see, e.g., [103,20])

$$\begin{aligned} I_5 &= \frac{1}{\pi\alpha'} \int d^2\sigma [(G_{MN} + B_{MN})(X)\partial X^M\bar{\partial}X^N + \mathcal{R}\phi(X)] \\ &= \frac{1}{\pi\alpha'} \int d^2\sigma [(\hat{G}_{\mu\nu} + B_{\mu\nu})(x)\partial x^\mu\bar{\partial}x^\nu + e^{2\sigma(x)}[\partial y + \mathcal{A}_\mu(x)\partial x^\mu][\bar{\partial}y + \mathcal{A}_\nu(x)\bar{\partial}x^\nu] \\ &\quad + \mathcal{B}_\mu(x)(\partial x^\mu\bar{\partial}y - \bar{\partial}x^\mu\partial y) + \mathcal{R}\phi(x)] , \end{aligned} \quad (7.1)$$

¹⁶ The models with $n = 1$ and $n = -1$ are related by the analytic continuation $u \rightarrow iu$, $v \rightarrow -iv$, suggesting that formal solutions of the corresponding CFT’s may be related. This does not imply, of course, the equivalence of the two models once the periodicity condition on the spatial part of u and v is imposed.

where $X^M = (x^\mu, x^5)$, $x^\mu = (t, x^i, x^3)$, $x^5 \equiv y$ and

$$\hat{G}_{\mu\nu} \equiv G_{\mu\nu} - G_{55}\mathcal{A}_\mu\mathcal{A}_\nu, \quad G_{55} \equiv e^{2\sigma}, \quad \mathcal{A}_\mu \equiv G^{55}G_{\mu 5}, \quad \mathcal{B}_\mu \equiv B_{\mu 5}. \quad (7.2)$$

These definitions of fields guarantee manifest gauge invariance under $y' = y + \epsilon$, $\mathcal{A}'_\mu = \mathcal{A}_\mu - \partial_\mu \epsilon$, $B'_{\mu\nu} = B_{\mu\nu} + \epsilon(\partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu)$. Similar decomposition of the string action can be done in the case of several internal coordinates y^a . From the point of view of the low-energy effective field theory, this decomposition corresponds to starting with the $D = 5$ bosonic string effective action and assuming that one spatial dimension x^5 is compactified on a small circle. Dropping the massive Kaluza-Klein modes one then finds the following dimensionally reduced $D = 4$ action (see, e.g., [104])

$$S_4 = \int d^4x \sqrt{\hat{G}} e^{-2\Phi} \left[\hat{R} + 4(\partial_\mu \Phi)^2 - (\partial_\mu \sigma)^2 - \frac{1}{12}(\hat{H}_{\mu\nu\lambda})^2 - \frac{1}{4}e^{2\sigma}(F_{\mu\nu}(\mathcal{A}))^2 - \frac{1}{4}e^{-2\sigma}(F_{\mu\nu}(\mathcal{B}))^2 + O(\alpha') \right], \quad (7.3)$$

where, in addition to (7.2), we have defined

$$F_{\mu\nu}(\mathcal{A}) = 2\partial_{[\mu}\mathcal{A}_{\nu]}, \quad F_{\mu\nu}(\mathcal{B}) = 2\partial_{[\mu}\mathcal{B}_{\nu]}, \quad (7.4)$$

$$\hat{H}_{\lambda\mu\nu} = 3\partial_{[\lambda}B_{\mu\nu]} - 3\mathcal{A}_{[\lambda}F_{\mu\nu]}(\mathcal{B}), \quad \Phi = \phi - \frac{1}{2}\sigma.$$

The σ -model (7.1) duality transformation in the y -isometry direction induces the following target space transformation

$$\mathcal{A} \rightarrow \pm \mathcal{B}, \quad \mathcal{B} \rightarrow \pm \mathcal{A}, \quad \sigma \rightarrow -\sigma, \quad \phi \rightarrow \phi - \sigma, \quad (7.5)$$

$$\hat{G}_{\mu\nu} \rightarrow \hat{G}_{\mu\nu}, \quad B_{\mu\nu} \rightarrow B_{\mu\nu} + \mathcal{A}_\mu \mathcal{B}_\nu - \mathcal{B}_\mu \mathcal{A}_\nu, \quad \hat{H}_{\mu\nu\lambda} \rightarrow \hat{H}_{\mu\nu\lambda},$$

which is obviously an invariance of the action (7.3).

Given a conformal $D = 5$ σ -model and representing it in the form (7.1) one can read off the expressions for the corresponding $D = 4$ background fields which are then bound to represent a solution of the effective action (7.3). The string solution is, of course, described by the full $D = 5$ action (7.1). Since y was assumed to be a compact isometry direction of the $D = 5$ model, dual $D = 4$ backgrounds related by (7.5) represent the same string solution (CFT).

In certain special cases the nontrivial part of the action (7.3) can be expressed in terms of only one scalar and one vector and takes the familiar form (here we use the Einstein frame)

$$S_4 = \int d^4x \sqrt{\hat{G}_E} \left[\hat{R}_E - \frac{1}{2}(\partial_\mu \psi)^2 - \frac{1}{4}e^{-a\psi}(F_{\mu\nu})^2 + O(\alpha') \right]. \quad (7.6)$$

For example, if one sets $\phi = 0$ and $H_{MNK} = 0$ in the $D = 5$ action one obtains (7.6) with $\psi = -a\sigma$ and $a = \sqrt{3}$. This is the standard Kaluza-Klein reduction of the Einstein action. Another possibility is to set $\sigma = 0$ ($G_{55} = 1$), $\hat{H}_{\mu\nu\lambda} = 0$ and either the two vector fields proportional to each other, or let one of them vanish. This case corresponds to (7.6) with $\psi = \Phi$ and $a = 1$.

7.2. Exact $D = 4$ extreme black hole solutions

Classical string theory, when considered from the effective action point of view, is different from the Einstein theory at least in two respects: (i) its action contains terms of all orders in α' expansion, i.e. it is a one-parameter ‘deformation’ of the Einstein theory, and (ii) the action depends on other ‘massless’ fields which should be treated on an equal footing with the metric. These differences are reflected in the structure of black hole solutions. For example, the Schwarzschild solution is necessarily modified by α' corrections with dilaton becoming non-constant at the next to the leading order [105]. Also, the presence of the dilaton coupling implies the existence of new leading-order solutions describing charged black holes [84,85,86].

Let us summarize what is known about *exact* black hole solutions. Exact black hole solutions have been constructed in two [30] and three [106] dimensions using (gauged) WZW models based on the group $SL(2, R)$. As for higher dimensions, there are coset CFT or gauged WZW constructions of just the ‘throat’ regions of some four dimensional extreme magnetically charged dilatonic black holes [72,107,108]¹⁷ but no connection between complete asymptotically flat solutions and coset models was found. It appears that while non-extreme (and some non-supersymmetric extreme [110]) leading-order charged black hole solutions are necessarily modified by α' -corrections, certain extreme magnetic and electric black hole leading-order solutions are, in fact, exact. The magnetic and electric cases turn out to be very different and so should be discussed separately.

(I) Extreme magnetic solutions:

(1a) The extremal limit of a black five-brane [111] is an exact superstring solution [47], which upon dimensional reduction gives an exact five dimensional extreme black hole. This black hole has a magnetic type of charge associated with the antisymmetric tensor field.

(1b) One can also dimensionally reduce the exact solution of [47] down to four dimensions [112] to obtain an extreme magnetically charged Kaluza-Klein ($a = \sqrt{3}$) black hole [113] (see also [49]). The $D = 5$ string σ -model corresponding to this ‘ H -monopole’ reduction is

$$L = -\partial t \bar{\partial} t + F^{-1}(x)(\partial x^i \bar{\partial} x_i + \partial y \bar{\partial} y) + A_i(x)(\partial y \bar{\partial} x^i - \bar{\partial} y \partial x^i) + \mathcal{R}\phi(x) , \quad (7.7)$$

where $y = x^5$, $i = 1, 2, 3$, $r^2 = x_i x^i$, and

$$e^{2(\phi - \phi_0)} = e^{2\sigma} = F^{-1} , \quad F^{-1} = 1 + \frac{M}{r} , \quad (7.8)$$

$$\mathcal{A}_i = 0 , \quad \mathcal{B}_i = -A_i = \text{monopole} , \quad F_{ij} = H_{ij5} = M \epsilon_{ijk} \frac{x^k}{r^3} ,$$

(and $H_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma\sigma} \partial^\sigma e^{-2\phi}$, $\alpha, \beta, \dots = 1, 2, 3, 5$). More precisely, this background is T -dual (in the sense of (7.5)) to the standard extreme Kaluza-Klein black hole solution [84] which has constant dilaton. Its (1,1) supersymmetric extension is an exact conformal model having (4,4) worldsheet supersymmetry. In the heterotic case, we achieve the same by embedding into the superstring, i.e. by adding $SU(2)$ internal gauge field V background equal to the generalised connection with torsion ω_+ . Without embedding (for zero V) one

¹⁷ The fact that the form of the throat solution is unchanged under string α' corrections was noticed earlier in [109].

still has (4,0) supersymmetry, but it seems that finiteness is spoiled by anomaly (which is absent in the similar electric case model [20], see below).

(1c) The string σ -model corresponding to the embedding of the dilatonic ($a = 1$) extreme magnetic $D = 4$ black hole into $D = 5$ [114] can be represented as

$$L = -\partial t \bar{\partial} t + F^2(x) \partial x^i \bar{\partial} x_i \quad (7.9)$$

$$+ [\partial y + A_i(x) \partial x^i] [\bar{\partial} y + A_i(x) \bar{\partial} x^i] + A_i(\partial y \bar{\partial} x^i - \bar{\partial} y \partial x^i) + \mathcal{R}\phi(x) .$$

Here the modulus σ is constant (so that it is self-dual in the sense of (7.5)) and thus $\mathcal{A}_i = -\mathcal{B}_i = A_i$, while F , A_i and ϕ are the same as in (7.8). The (1,0) supersymmetric version of this model is actually (4,0) supersymmetric [114] (the corresponding background breaks ‘one-half’ of space-time supersymmetry when embedded into $D = 10$ supergravity). The same model was considered by starting directly from $D = 10$ (and was generalised to multi-center case) in [115] where it was pointed out that it becomes (4,1) supersymmetric once one embeds the heterotic string solution into the superstring theory, i.e. introduces also the gauge field background $V = \omega_+$. Then using the result of [116] one may argue that this model is an exact heterotic string solution.

(II) Extreme electric solutions:

(2a) The extreme electric ‘Kaluza-Klein’ ($a = \sqrt{3}$) $D = 4$ black hole [84] can be obtained by dimensional reduction either of $D = 5$ plane wave background [84] or of the dual $D = 5$ fundamental string model [117]. Since these two $D = 5$ backgrounds are exact (bosonic) string solutions [21], this black hole solution is exact.

(2b) The extreme electric dilatonic ($a = 1$) $D = 4$ black hole [85,86] can be obtained [36] by dimensional reduction from a generalised $D = 5$ fundamental string model (found as a leading order solution in [83,102]) which is also an exact string solution [36,20].

(3c) The analogs and generalisations of the Israel-Wilson-Perjés (IWP) [118] solution of the pure Einstein-Maxwell theory, including as special cases Majumdar-Papapetrou-type solution (a collection of extremal electric dilatonic black holes) and an extremal electric Taub-NUT-type solution are dimensional reductions of the generalised $D = 5$ chiral null model [20] and as such are exact string solutions (some of these backgrounds were originally found as leading-order string solutions in [87,88,89]).

Below we shall discuss in more detail the extreme electric black hole solutions and their generalisations. Let us first note several important differences between the above magnetic and electric solutions. The $D = 5, 4$ extreme magnetically charged solutions are products of a time-like line and a euclidean solution, while the electric solutions are non-trivial in the time-like direction. Second, these magnetic extreme black holes were shown to be exact solutions only in the superstring (or heterotic string) theory. Extended (4, n) supersymmetry (on the world-sheet, related to that on space-time) played a key role in the arguments that there are no α' -corrections to the leading-order backgrounds. The solutions we discuss below, like the $D = 2, 3$ examples, are exact already in the bosonic string theory (they also have, of course, exact analogs in the superstring and heterotic string theories [20]).

Let us start with a simple $D = 5$ plane wave model (see (4.9), (4.22), (4.11))

$$L = \partial u \bar{\partial} v + K(x) \partial u \bar{\partial} u + \partial x_i \bar{\partial} x^i + \mathcal{R}\phi_0 , \quad K = 1 + \frac{M}{r} . \quad (7.10)$$

With given K this is a conformal model (assuming $r > 0$; one needs a δ -function source at the origin to satisfy the Laplace equation at all points). Identifying u with the internal coordinate y and v with $2t$ we find that (7.10) corresponds to the following $D = 4$

background (cf.(7.1))

$$ds_4^2 = -F(r)dt^2 + dx_i dx^i, \quad e^{2\sigma} = F^{-1}(r), \quad \mathcal{A}_t = F(r), \quad (7.11)$$

$$F \equiv K^{-1} = \left(1 + \frac{M}{r}\right)^{-1}.$$

This is just the extreme electrically charged Kaluza-Klein black hole [84,113], which can thus be viewed as an exact string solution [36].¹⁸ This exact electric solution is related to the exact magnetic solution (7.8) by S -duality.

Similar background can be obtained by using another conformal σ -model which describes the $D = 5$ fundamental string background

$$L = F(x)\partial u \bar{\partial} v + \partial x_i \bar{\partial} x^i + \mathcal{R}\phi(x), \quad (7.12)$$

$$e^{2(\phi-\phi_0)} = F(r) = \left(1 + \frac{M}{r}\right)^{-1}. \quad (7.13)$$

With $u = y - t$, $v = y + t$ this leads again to the same 4-metric as in (7.11) and with $\mathcal{A}_t \rightarrow -\mathcal{B}_t$, $\sigma \rightarrow -\sigma$, $\phi \rightarrow \phi - \sigma$. This is consistent with the duality transformation rule (7.5). Indeed, (7.12) is y -dual to (7.10) (i.e. represents the same CFT) with u and v in (7.10) being indeed $u = \tilde{y}$, $v = 2t$. Note that (7.12) is conformal (cf.(5.3)) without need for a source at the origin. Thus the $a = \sqrt{3}$ extreme electric black hole background can be viewed as coming from an exact solution without sources.

This is an example of what may be a general rule: for spherically symmetric charged solutions the duality relates ‘fundamental’ (supported by a source) and ‘solitonic’ (sourceless) solutions. It appears that the usual duality transformation may map solutions of low-energy effective equations into solutions only when the metric is invertible. Solutions with a singularity at $r = 0$ can be mapped into configurations requiring sources.

Starting with the following conformal model which is a generalisation (6.9) ($n = 1$) of the fundamental string model (7.12)

$$L = F(x)\partial u \bar{\partial} v + \partial u \partial u + \partial x_i \bar{\partial} x^i + \mathcal{R}\phi(x), \quad (7.14)$$

and setting $u = y$, $v = 2t$ one finds upon dimensional reduction to $D = 4$

$$ds_4^2 = -F^2(r)dt^2 + dx_i dx^i, \quad \phi(x) = \phi_0 + \frac{1}{2}\ln F(r), \quad (7.15)$$

$$\mathcal{A}_t = -\mathcal{B}_t = F(r), \quad \mathcal{A}_i = \mathcal{B}_i = \sigma = \hat{H}_{\lambda\mu\nu} = 0.$$

¹⁸ It should be noted that while the choice of a particular direction $y = bu + cv$ of dimensional reduction does not matter for the purpose of deriving the dimensionally reduced $D = 4$ background (7.11) (different reductions will give gauge-equivalent backgrounds) the corresponding string models (7.10) with different choices of the definition of the compact coordinate y are, in general, inequivalent. For example, the choices of $u = y - t$, $v = y + t$ and $u = y'$, $v = 2t'$ are not equivalent since the relations $t = t' - \frac{1}{2}y'$, $y = \frac{1}{2}y' + t'$ are not consistent with having y and y' both compact while t and t' both noncompact.

This is the extremal limit of the dilatonic $a = 1$ electric black hole [85,86]. Again, S -duality relates this solution to the exact magnetic solution (7.9). Thus the (Kaluza-Klein embedded) $U(1)$ magnetic solutions of the heterotic string are S -dual to the extreme electric solutions described by chiral null models (see also [119]).

The $r = 0$ singularity of this background (and also of the fundamental string one (7.12)) is probably absent at the level of the corresponding CFT (as is suggested by the form of the tachyon equation, see also [120]). The model (7.14) is ‘self-dual’ so the problem with sources in the dual version does not appear here.

Exactness of the source-free extreme black hole solutions (supersymmetric in the magnetic case and Kaluza-Klein embedded in the bosonic, superstring or heterotic string in the electric case) supports their solitonic interpretation.

7.3. More general $D = 4$ solutions obtained from $D = 5$ chiral null models

Further generalisation is provided by the $D = 5$ chiral null model

$$L = F(x)\partial u[\bar{\partial}v + K(x)\bar{\partial}u + 2A_i(x)\bar{\partial}x^i] + \partial x_i\bar{\partial}x^i + \mathcal{R}\phi(x) , \quad (7.16)$$

which is conformally invariant if the functions F , K , ϕ and A_i of the three coordinates x^i satisfy (6.6),(6.5),(6.7), i.e.

$$\partial^2 F^{-1} = 0 , \quad \partial^2 K = 0 , \quad \partial_i F^{ij} = 0 , \quad \phi = \phi_0 + \frac{1}{2}\ln F , \quad (7.17)$$

$$\epsilon^{ijk}\partial_j A_k = \partial^i T , \quad \partial^2 T = 0 . \quad (7.18)$$

The model is thus parametrised by 3 harmonic functions. In particular, one can choose

$$F^{-1} = 1 + \sum_{k=1}^N \frac{M_k}{|x - x_k|} , \quad K = 1 + \sum_{k=1}^N \frac{m_k}{|x - x_k|} , \quad T = \sum_{k=1}^N \frac{q_k}{|x - x_k|} . \quad (7.19)$$

With $u = y$, $v = 2t$ the corresponding class of exact $D = 4$ backgrounds is [20]

$$ds_4^2 = -F(x)K^{-1}(x)[dt + A_i(x)dx^i]^2 + dx_i dx^i , \quad (7.20)$$

$$\mathcal{A}_t = K^{-1}(x) , \quad \mathcal{A}_i = K^{-1}(x)A_i(x) , \quad \mathcal{B}_t = -F(x) , \quad \mathcal{B}_i = -F(x)A_i(x) ,$$

$$\sigma = \frac{1}{2}\ln[F(x)K(x)] , \quad \phi = \phi_0 + \frac{1}{2}\ln F(x) , \quad \hat{H}_{\lambda\mu\nu} = -6\mathcal{A}_{[\lambda}\partial_\mu\mathcal{B}_{\nu]} .$$

This class is ‘self-dual’ since under the duality $F \rightarrow K^{-1}$ (cf.(7.5)). The above extreme electric solutions (7.11) and (7.15) are obviously included as special cases.

The case of $K = F^{-1}$ is of a particular interest. Since $\sigma = 0$ and the two gauge fields differ only by a sign, these backgrounds are solutions of the equations following from (7.3),(7.6) with $a = 1$. These are precisely the $D = 4$ dilatonic IWP backgrounds found as the leading-order solutions in [87,88,89]. Adding a nonzero A_i to the solution (7.15) by setting $T = q/r$ has the effect of adding a NUT charge. The result is the extremal electrically charged dilatonic Taub-NUT solution. One can also get linear superposition of an arbitrary number of solutions of this type by taking F and T to be ‘multicenter’ harmonic functions as in (7.19). The solutions of the Laplace equations in (7.17) which are singular on circles, rather than points correspond to adding angular momentum. The

solutions with generic K and thus different gauge fields ($\mathcal{A}_\mu \neq \pm \mathcal{B}_\mu$) were first found in [20]. Some (IWP) solutions obtained from higher ($D = 10$) dimensional chiral null models were considered in [121, 119]. Leading-order Kaluza-Klein black hole solutions related to the above solutions by the electro-magnetic duality were discussed in [122].

Let us note again that even solving the leading order string analog of the Einstein's equations can be rather complicated when the dilaton and antisymmetric tensor are non-trivial. By choosing an ansatz at the level of the string world sheet action which yields simple equations for the σ -model β -functions, one can easily find new solutions of the leading order equations. The chiral null models provide an example of this. Another example is magnetic flux tube solutions found in [22] (see Section 8).

The simplest example of the solution with $K \neq F^{-1}$ is found in the spherically symmetric case

$$F^{-1} = 1 + \frac{M}{r}, \quad K = 1 - \frac{M}{r}, \quad A_i = 0, \quad (7.21)$$

corresponding to the $D = 5$ chiral null model (6.9) with $n = -1$ (and redefined v) which, like (7.14) describes $D = 5$ fundamental string with linear momentum flowing along the string. The resulting $D = 4$ background (7.20)

$$ds_4^2 = -(1 - \frac{M^2}{r^2})^{-1} dt^2 + dx_i dx^i, \quad (7.22)$$

$$\mathcal{A}_t = (1 - \frac{M}{r})^{-1}, \quad \mathcal{B}_t = -(1 + \frac{M}{r})^{-1}, \quad \mathcal{B}_i = \mathcal{A}_i = 0, \quad \hat{H}_{\lambda\mu\nu} = 0,$$

$$\sigma = \frac{1}{2} \ln[(1 - \frac{M}{r})/(1 + \frac{M}{r})], \quad \phi = \phi_0 - \frac{1}{2} \ln(1 + \frac{M}{r}),$$

$$\varphi = 2\phi - \sigma = 2\phi_0 - \frac{1}{2} \ln(1 - \frac{M^2}{r^2}),$$

was interpreted in [119] as a 'massless' extreme electric black hole (note that there is no $1/r$ -term in the metric coefficient; M here plays the role of an electric charge and not of the ADM mass which vanishes). Note that under the duality (7.5) this background is transformed into itself with $M \rightarrow -M$. The close similarity of the two $D = 5$ chiral null models ((6.9) with $n = 1$ and $n = -1$) corresponding to the $a = 1$ extreme electric black hole (7.15) and the 'massless' black hole (7.22) suggests that the two $D = 4$ solutions may be in certain sense 'complementary'.

Another interesting special case is $F = K = 1$. The corresponding model is a special case of (4.14) (cf. (7.16))

$$L = \partial u \bar{\partial} v + 2A_i(x) \bar{\partial} x^i + \partial x_i \bar{\partial} x^i + \mathcal{R}\phi(x), \quad (7.23)$$

where A_i satisfies (7.18). Particular solutions for A_i are:

(1) $F_{ij} = p\epsilon_{ijk}x^k/r^3$, corresponding to extreme magnetically charged dilatonic Taub-NUT solution with zero mass;

(2) $A_i = -\frac{1}{2}F_{ij}x^j$, $F_{ij} = \text{const}$ or $F_{i3} = 0$, $F_{ij} = \beta\epsilon_{ij}$, $i, j = 1, 2$, describing a 'rotating magnetic universe' or 'constant magnetic field solution' [20, 51] (see (7.20))

$$ds_4^2 = -(dt + \frac{1}{2}\beta\rho^2 d\varphi)^2 + d\rho^2 + \rho^2 d\varphi^2 + dx_3^2, \quad (7.24)$$

$$\mathcal{A} = -\mathcal{B} = A = \frac{1}{2}\beta\rho^2 d\varphi, \quad \sigma = \phi - \phi_0 = 0, \quad B = \frac{1}{2}\beta\rho^2 d\varphi \wedge dt, \quad \hat{H}_{tij} = F_{ij}.$$

The stationary metric (7.24) is the metric of a homogeneous space – the direct product of the group space of the Heisenberg group and x_3 -direction. This solution has a non-trivial antisymmetric tensor (and constant dilaton) and thus, in contrast to the dilatonic Melvin solution [123,85], has no analog in the Einstein-Maxwell theory.

We shall discuss this and other uniform magnetic field solutions in Sections 8,9.

7.4. Charged fundamental string solutions

Another example of an exact solution obtained by dimensional reduction is the charged fundamental string background found at the leading order level in [82,83]. Starting with the general chiral null model in $D+N$ dimensions, requiring that all the fields are independent of u and N of the transverse dimensions y^a and assuming that the vector coupling has only y^a -components we obtain

$$L = F(x)\partial u\bar{\partial}v + \tilde{K}(x)\partial u\bar{\partial}u + \partial x_i\bar{\partial}x^i + 2\tilde{A}_a(x)\partial u\bar{\partial}y^a + \partial y_a\bar{\partial}y^a + \mathcal{R}\phi(x). \quad (7.25)$$

This model is conformal to all orders provided F , $K \equiv F^{-1}\tilde{K}$, $A_a \equiv F^{-1}\tilde{A}_a$ and ϕ satisfy (6.8). Looking for solutions which are rotationally symmetric in $D-2$ coordinates x^i and solving the Laplace equations we can put the functions F, K, A_a in the form¹⁹

$$F^{-1} = 1 + \frac{M}{r^{D-4}}, \quad \phi = \phi_0 + \frac{1}{2}\ln F(r), \quad K = c + \frac{P}{r^{D-4}}, \quad A_a = \frac{Q_a}{r^{D-4}}. \quad (7.26)$$

Shifting v we can in general replace \tilde{K} in (7.25) by a constant. To reinterpret (7.25) as a D -dimensional model coupled to N internal coordinates we should rewrite it in the form (7.1)

$$L = F(r)\partial u\bar{\partial}v + [\tilde{K} - (\tilde{A}_a)^2](r)\partial u\bar{\partial}u + \partial x_i\bar{\partial}x^i + \mathcal{R}\phi(r) \\ + \tilde{A}_a(r)(\partial u\bar{\partial}y^a - \partial y^a\bar{\partial}u) + [\partial y^a + \tilde{A}^a(r)\partial u][\bar{\partial}y_a + \tilde{A}_a(r)\bar{\partial}u]. \quad (7.27)$$

The first four terms give the D -dimensional space-time metric, antisymmetric tensor and dilaton while the last two indicate the presence of the two equal vector field backgrounds (see (7.1)). Note that since $G_{ab} = \delta_{ab}$, the modulus field is constant (so that the lower dimensional dilaton is the same as the higher dimensional one). In the case of just one internal dimension we get one abelian vector field with non-vanishing u -component and the resulting background becomes that of the charged fundamental string of [82,83]. Again, this model is conformal without need to introduce sources suggesting a solitonic interpretation of the corresponding background.

8. Other examples of exact solutions: magnetic flux tube backgrounds

One may look for other exact conformal σ -models by trying to generalise the chiral null model Lagrangian (6.2). One possibility is to replace its flat transverse part by a conformal model just as this was done in the particular cases of K - and F - models (4.21), (5.1). If the total dimension is $D = 4$ that means again to replace the flat 2-space by the euclidean

¹⁹ For $Q_a = 0$ we get not just the fundamental string solution of [35] but its modification (7.14) which corresponds to momentum running along the string.

black hole $SL(2, R)/U(1)$ model (or $SU(2)/U(1)$ background). Examples of such exact $D = 4$ ‘hybrid’ solutions with $A_i = 0$ in (6.2) were already given in (4.25), (4.26), (5.10). Similar solutions exist when the vector coupling function A_i is non-vanishing and satisfies the curved transverse space analogue of (6.5), $\partial_i(e^{-2\phi}\sqrt{G}\mathcal{F}^{ij}) = 0$. In particular, the generalisation of the constant magnetic field solution (7.24) is [22]

$$ds_4^2 = -(dt + \frac{1}{2}\beta b^{-2}\tanh^2 b\rho \, d\varphi)^2 + d\rho^2 + b^{-2}\tanh^2 b\rho \, d\varphi^2 + dx_3^2, \quad (8.1)$$

$$\mathcal{A} = -\mathcal{B} = A = \frac{1}{2}\beta b^{-2}\tanh^2 b\rho \, d\varphi,$$

$$B = \frac{1}{2}\beta b^{-2}\tanh^2 b\rho \, d\varphi \wedge dt, \quad e^{\phi-\phi_0} = (\cosh b\rho)^{-1}, \quad \sigma = 0.$$

The corresponding $D = 5$ σ -model (cf. (7.23))

$$L = \partial u \bar{\partial} v + \beta b^{-2}\tanh^2 b\rho \, \partial u \bar{\partial} \varphi + \partial \rho \bar{\partial} \rho + b^{-2}\tanh^2 b\rho \, \partial \varphi \bar{\partial} \varphi + \mathcal{R}\phi(\rho), \quad (8.2)$$

is conformal to all orders in the leading-order scheme. The constant b ($\alpha' b^2 = 1/k$) is fixed by the condition of the vanishing of the total central charge, $2 + 3k/(k-2) + N - 26 = 0$, where N is a number of extra free bosonic dimensions. The solution (8.1) reduces to (7.24) in the limit $b \rightarrow 0$ (i.e. $N \rightarrow 22$). The introduction of the parameter b changes the large ρ behaviour of the background fields (while for small ρ the form of the fields remains the same as for $b = 0$).

One may also try to extend the chiral null model by adding extra couplings, e.g., $2\tilde{A}'_i(x)\partial x^i \bar{\partial} v$. In general, such models will no longer be conformal invariant to all orders if only the leading-order β -function conditions are satisfied [20]. There exists, however, a special case when the above term is supplemented by a particular extra term in the ‘transverse’ part of the action,

$$L = F(x) [\partial u + 2A'_i(x)\partial x^i] [\bar{\partial} v + 2A_i(x)\bar{\partial} x^i] + \partial x_i \bar{\partial} x^i + \mathcal{R}\phi(x). \quad (8.3)$$

This ‘generalised F -model’ can be obtained by u -duality from the ‘non-chiral’ generalization of the plane-wave K -model (4.20) with two non-vanishing vector couplings (the relation between the functions in (8.3) and in (4.20) is: $F = K^{-1}$, $A'_i = -\bar{A}_i$, $A_i = \bar{A}_i$, $\phi = \phi_0 + \frac{1}{2}\ln F$). When A_i and A'_i have constant field strengths, the theory (8.3) can be shown to be conformally invariant to all loop orders, provided [20] (cf. (4.17), (4.18))²⁰

$$\frac{1}{2}\partial^2 F^{-1} + \mathcal{F}^{ij}\mathcal{F}'_{ij} = 0, \quad \phi = \phi_0 + \frac{1}{2}\ln F. \quad (8.4)$$

A particular $D = 5$ solution is described by [22] ($\alpha, \beta = \text{const}$)

$$L = F(\rho)(\partial u - \alpha\rho^2\partial\varphi)(\bar{\partial} v + \beta\rho^2\bar{\partial}\varphi) + \partial\rho\bar{\partial}\rho + \rho^2\partial\varphi\bar{\partial}\varphi + \partial x_3\bar{\partial} x_3, \quad (8.5)$$

$$+ \mathcal{R}(\phi_0 + \frac{1}{2}\ln F), \quad F^{-1} = F_0^{-1} + \alpha\beta\rho^2,$$

²⁰ While the model obtained by adding $\tilde{K}(x)\partial u \bar{\partial} u$ to (8.3) is not exactly conformal for any \tilde{K} there exists a generalisation of this exact solution with one of the field strengths being constant and another being an arbitrary (e.g. monopole in $D = 3$) solution of the Maxwell equation.

where F_0^{-1} is a solution of the homogeneous $D = 3$ (ρ, φ, x_3) Laplace equation, e.g. (cf. (4.19))

$$F^{-1}(\rho) = 1 - \mu \ln \frac{\rho}{\rho_0} + \alpha\beta\rho^2 , \quad (8.6)$$

or

$$F^{-1}(\rho, x_3) = 1 + \frac{M}{r} + \alpha\beta\rho^2 , \quad r^2 \equiv \rho^2 + x_3^2 , \quad (8.7)$$

or

$$F^{-1}(r) = 1 + \frac{M}{r} + \frac{2}{3}\alpha\beta r^2 . \quad (8.8)$$

Dimensional reduction along the $y = \frac{1}{2}(u + v)$ -direction (see (7.1)) leads to (electro)magnetic flux tube backgrounds [22]

$$ds_4^2 = -F(\rho)[dt + \frac{1}{2}(\alpha + \beta)\rho^2 d\varphi]^2 + d\rho^2 + \rho^2 d\varphi^2 + dx_3^2 , \quad (8.9)$$

$$B = \frac{1}{2}(\beta - \alpha)F(\rho)\rho^2 d\varphi \wedge dt , \quad e^{2\sigma} = e^{2(\phi - \phi_0)} = F(\rho) , \quad (8.10)$$

$$\mathcal{A} = \frac{1}{2}(\beta - \alpha)\rho^2 d\varphi , \quad \mathcal{B} = -\frac{1}{2}F(\rho)\rho^2[(\alpha + \beta)d\varphi - 2\alpha\beta dt] , \quad (8.11)$$

with the ‘constant magnetic field’ solution (7.24) as a special $F_0 = 1$, $\alpha = 0$ case. The vector field \mathcal{A} has constant magnetic field strength while \mathcal{B} has both magnetic and electric components. The curvature is non-singular for $\alpha\beta \geq 0$.

The background (8.9),(8.10),(8.11) with F given by (8.7) represent a black-hole type configuration in an external electromagnetic field, namely, a generalisation to the case of $\alpha \neq 0$ of the solution (7.16),(7.20) (with $K = 1$, $F = 1/(1 + M/r)$) which was an extension ($\beta \neq 0$) of the extremal Kaluza-Klein ($a = \sqrt{3}$) black hole.²¹

It is possible also obtain a more general globally inequivalent conformal model by replacing $\varphi \in (0, 2\pi)$ in (8.5) by $\varphi' = \varphi + qy$, $y \equiv \frac{1}{2}(u + v)$ [22]. Reducing in y direction in the case of $F_0 = 1$ one finds a 3-parameter (α, β, q) class of exact $D = 4$ axially symmetric flux tube solutions [22] which generalise the constant magnetic field solution (7.24) and the dilatonic ($a = 1$) and Kaluza-Klein ($a = \sqrt{3}$) Melvin solutions [85,124] which in the string σ -model frame have the form (cf. (7.24))

$$a = 1 : \quad ds_4^2 = -dt^2 + d\rho^2 + F^2(\rho)\rho^2 d\varphi^2 + dx_3^2 , \quad (8.12)$$

$$\mathcal{A} = -\mathcal{B} = \beta F(\rho)\rho^2 d\varphi , \quad B = 0 ,$$

$$\sigma = 0 , \quad e^{2(\phi - \phi_0)} = F = (1 + \beta^2 \rho^2)^{-1} ,$$

²¹ At the same time, the solutions (8.5) with F given by (8.7) do not include a generalization of the extreme electric dilatonic ($a = 1$) black hole since the model (8.5) does not contain the term $K\partial u \bar{\partial} u$ (or $K\partial v \bar{\partial} v$) which is necessary in order to obtain this black hole background by dimensional reduction [36]. In fact, for $\alpha\beta \neq 0$ such a term (or, e.g., its ‘gauge-invariant’ generalization $K(\partial u - 2\alpha A)(\bar{\partial} u - 2\alpha \bar{A})$) cannot be added to (8.5) without spoiling the conformal invariance of the model.

and

$$a = \sqrt{3} : \quad ds_4^2 = -dt^2 + d\rho^2 + \tilde{F}(\rho)\rho^2 d\varphi^2 + dx_3^2, \quad (8.13)$$

$$\mathcal{A} = q\tilde{F}(\rho)\rho^2 d\varphi, \quad \mathcal{B} = 0, \quad B = 0,$$

$$\phi = \phi_0, \quad e^{2\sigma} = \tilde{F}^{-1} = 1 + q^2\rho^2.$$

The resulting expressions for the $D = 4$ background fields (which solve, in particular, the leading-order equations following from (7.3)) are given by [22]

$$ds_4^2 = -dt^2 + F(\rho)\rho^2(d\varphi - \alpha dt)(d\varphi - \beta dt) \quad (8.14)$$

$$- \frac{1}{4}F(\rho)\tilde{F}(\rho)\rho^4[(\alpha - \beta - 2q)d\varphi + q(\alpha + \beta)dt]^2 + d\rho^2 + dx_3^2,$$

$$\mathcal{A} = -\frac{1}{2}\tilde{F}(\rho)\rho^2[(\alpha - \beta - 2q)d\varphi + q(\alpha + \beta)dt], \quad (8.15)$$

$$\mathcal{B} = -\frac{1}{2}F(\rho)\rho^2[(\alpha + \beta)d\varphi - (2\alpha\beta + q\alpha - q\beta)dt],$$

$$e^{2(\phi - \phi_0)} = F(\rho), \quad e^{2\sigma} = \frac{F(\rho)}{\tilde{F}(\rho)}, \quad B = -\frac{1}{2}(\alpha - \beta)F(\rho)\rho^2 d\varphi \wedge dt, \quad (8.16)$$

$$F(\rho) \equiv \frac{1}{1 + \alpha\beta\rho^2}, \quad \tilde{F}(\rho) \equiv \frac{1}{1 + q(q + \beta - \alpha)\rho^2}. \quad (8.17)$$

The metric is stationary and, in general, describes a rotating ‘universe’. For generic values of the parameters the abelian gauge fields contain both magnetic and electric components with the former being more ‘fundamental’ (there are no solutions when both of the gauge fields are pure electric). For simplicity we shall call these solutions ‘magnetic flux tube backgrounds’. The uniform pure magnetic field solutions (7.24), (8.12) and (8.13) are the following special cases: $q = \alpha = 0, \beta \neq 0$; $\alpha = \beta = q \neq 0$, and $\alpha = \beta = 0, q \neq 0$. In addition to the $q = 0$ subclass of magnetic backgrounds (8.9), (8.10), (8.11) there are two other special subclasses: $\alpha = q$ (stationary metric, non-zero $B_{\mu\nu}$, zero σ) and $\alpha = \beta$ (static metric, zero $B_{\mu\nu}$, non-zero σ).

It is interesting to note that performing the electromagnetic S -duality on the magnetic Melvin backgrounds (for any value of a) one finds leading-order solutions with *constant* electric field. These electric backgrounds do not, however, represent the exact classical string solutions (i.e. they are corrected at each order in α'), and thus, in contrast to the magnetic Melvin case, the conformal σ -model which corresponds to them is not explicitly known.²²

Though the magnetic flux tube backgrounds (8.14)–(8.17) look quite complicated the conformal string model corresponding to them can be solved explicitly. This will be discussed in the next section.

²² Let us mention also some other known exact magnetic solutions in string theory. String (electro)magnetic backgrounds obtained by dimensional reduction of $SU(1,1)$ WZW model were considered in [125]. An exact $D = 3$ monopole [126] magnetic background based on $SU(2)$ WZW model tensored with linear dilaton was discussed in [127]. Heterotic string models with magnetic monopole type or dyon type backgrounds were constructed in [72,108]. A conformal embedding of a monopole-type magnetic field into the gauge sector of the heterotic string theory which uses $SU(2)$ WZW model was studied in [128]. The Robinson-Bertotti solution has an exact string counterpart [107] which is a product of the two conformal theories: ‘(AdS)₂’ ($SL(2, R)/Z$ WZW) and ‘monopole’ ($SU(2)/Z_m$ WZW) [72] ones.

9. Exactly solvable string models corresponding to magnetic flux tube backgrounds

In contrast to generic chiral null models (e.g., to fundamental string and extreme electric black hole solutions) for which the corresponding CFT is not known (in particular, one does not know the exact form of the equations for tachyon and other string modes propagating in these backgrounds) the conformal string models which represent the magnetic flux tube backgrounds ((8.5) and its generalisations) can be defined and solved explicitly. Namely, one can find the quantum Hamiltonian expressed in terms of free oscillators, determine the string spectrum, compute the partition function, etc. [51,22]. These models appear to be simpler than coset CFT's corresponding to semisimple gauged WZW solutions.²³ For example, their unitarity is easy to demonstrate because of the existence of a light-cone gauge. They provide (along with non-semisimple coset models [37,39,38,40,93]) one of the first non-trivial solvable examples of a consistent string theory in curved spacetimes. Some features of these models (in particular, the presence of tachyonic instabilities in the spectrum and existence of critical values of magnetic field) are similar to those of the conformal model describing open string in a constant magnetic field [129,130,131,132].

Generalisations to the case of superstring and heterotic string theories are discussed in [51,133].

9.1. String actions and angular duality

As discussed in Section 7, to describe the coupling of a closed bosonic string to a magnetic gauge field background one introduces a compact internal coordinate y . Then the string models which correspond to the $D = 4$ magnetic backgrounds (7.24) ('constant magnetic field'), (8.12) ('dilaton $a = 1$ Melvin flux tube') and (8.13) ('Kaluza-Klein $a = \sqrt{3}$ Melvin flux tube') are represented by the following $D = 5$ σ -models [51,134,22]

$$L_{\alpha=q=0} = \partial u \bar{\partial} v + \beta \epsilon_{ij} x^i \bar{\partial} x^j \partial u + \partial x_i \bar{\partial} x^i + \partial x_3 \bar{\partial} x_3 + \mathcal{R} \phi_0 \quad (9.1)$$

$$= \partial u \bar{\partial} v + \beta \rho^2 \bar{\partial} \varphi \partial u + \partial \rho \bar{\partial} \rho + \rho^2 \partial \varphi \bar{\partial} \varphi + \partial x_3 \bar{\partial} x_3 + \mathcal{R} \phi_0 ,$$

$$L_{\alpha=\beta=q} = -\partial t \bar{\partial} t + \partial \rho \bar{\partial} \rho + F(\rho) \rho^2 (\partial \varphi + 2\beta \partial y) \bar{\partial} \varphi + \partial y \bar{\partial} y + \partial x_3 \bar{\partial} x_3 + \mathcal{R} \phi(\rho) , \quad (9.2)$$

$$e^{2(\phi-\phi_0)} = F(\rho) = (1 + \beta^2 \rho^2)^{-1} ,$$

$$L_{\alpha=\beta=0} = -\partial t \bar{\partial} t + \partial \rho \bar{\partial} \rho + \rho^2 (\partial \varphi + q \partial y) (\bar{\partial} \varphi + q \bar{\partial} y) + \partial y \bar{\partial} y + \partial x_3 \bar{\partial} x_3 + \mathcal{R} \phi_0 , \quad (9.3)$$

where $x_1 + ix_2 = \rho e^{i\varphi}$, $\varphi \in (0, 2\pi)$ and

$$u \equiv y - t , \quad v \equiv y + t , \quad y \in (0, 2\pi R) . \quad (9.4)$$

All three models have free-theory central charge. In the case of non-compact y , i.e. in the limit $R \rightarrow \infty$, they become equivalent to other known models. The constant field model (9.1) becomes the E_2^c WZW model of [37] (or 'plane wave' (4.8)) with the corresponding CFT discussed in [39,51,135]. The $a = 1$ Melvin model (9.2) with coordinates formally taken to be non-compact can be identified with a particular limit of $[SL(2, R) \times R]/R$

²³ For a review of solvable (super)string models based on semisimple coset CFT's see, e.g., [52,48].

gauged WZW (‘black string’) model [134]²⁴ or, equivalently, with the $E_2^c/U(1)$ coset theory [40]. The $R = \infty$ case of the $a = \sqrt{3}$ Melvin model (9.3) is identical to the flat space model after the redefinition of φ .

The solvability of these and more general 3-parameter (α, β, q) class of models can be explained using their relation via duality and formal coordinate shifts to flat models. Consider, for example, the σ -model which a direct product of $D = 2$ Minkowski space and $D = 2$ ‘dual 2-plane’

$$\tilde{I} = \frac{1}{\pi\alpha'} \int d^2\sigma [\partial u \bar{\partial} v + \partial \rho \bar{\partial} \rho + \rho^{-2} \partial \tilde{\varphi} \bar{\partial} \tilde{\varphi} + \mathcal{R}(\phi_0 - \ln \rho)] . \quad (9.5)$$

$\tilde{\varphi}$ should have period $2\pi\alpha'$ to preserve equivalence of the ‘dual 2-plane’ model to the flat 2-plane CFT [136], i.e. to the flat model²⁵

$$I = \frac{1}{\pi\alpha'} \int d^2\sigma (\partial u \bar{\partial} v + \partial \rho \bar{\partial} \rho + \rho^2 \partial \varphi \bar{\partial} \varphi + \mathcal{R}\phi_0) . \quad (9.6)$$

If we now make coordinate shifts and add a constant antisymmetric tensor term we obtain (α, β, q are free parameters of dimension cm^{-1})

$$\begin{aligned} I = \frac{1}{\pi\alpha'} \int d^2\sigma [(\partial u + \alpha \partial \tilde{\varphi})(\bar{\partial} v + \beta \bar{\partial} \tilde{\varphi}) + \partial \rho \bar{\partial} \rho + \rho^{-2} \partial \tilde{\varphi} \bar{\partial} \tilde{\varphi} \\ + \frac{1}{2} q [\partial(u + v) \bar{\partial} \tilde{\varphi} - \bar{\partial}(u + v) \partial \tilde{\varphi}] + \mathcal{R}(\phi_0 - \ln \rho)] . \end{aligned} \quad (9.7)$$

The two models (9.5) and (9.7) are of course ‘locally-equivalent’; in particular, (9.7) also solves the conformal invariance equations. However, if u and v are given by (9.4) the ‘shifted’ coordinates $u + \alpha \tilde{\varphi}$ and $v + \beta \tilde{\varphi}$ are not globally defined for generic α and β (since the periods of $y = \frac{1}{2}(u + v)$ and φ are different) and the torsion term is non-trivial for $q \neq 0$. As a result, the conformal field theories corresponding to (9.5) and (9.7) will not be equivalent. The $O(3, 3; R)$ duality transformation with *continuous* coefficients which relates the model (9.7) to a flat one (9.6) is not a symmetry of the flat CFT, i.e. leads to a new conformal model which, however, is simple enough to be explicitly solvable [51].

Starting with (9.7) and making the duality transformation in $\tilde{\varphi}$ one obtains a more complicated σ -model

$$\begin{aligned} L = F(\rho)(\partial u - \alpha \rho^2 \partial \varphi')(\bar{\partial} v + \beta \rho^2 \bar{\partial} \varphi') + \partial \rho \bar{\partial} \rho + \rho^2 \partial \varphi' \bar{\partial} \varphi' \\ + \partial x_3 \bar{\partial} x_3 + \mathcal{R}(\phi_0 + \frac{1}{2} \ln F) , \end{aligned} \quad (9.8)$$

²⁴ In this limit $k \rightarrow \infty$ and the mass and charge of ‘black string’ [67] vanish but simultaneous rescalings of coordinates give rise to a nontrivial model.

²⁵ The two models are equivalent in the sense of a relation of classical solutions and equality of the correlators of certain operators (e.g., $\partial \tilde{\varphi}$ and $\rho^2 \partial \varphi$) but the spectra of states are formally different (cf. also [31, 137]): the spectrum is continuous on 2-plane and discrete on dual 2-plane (with duality relating states with given orbital momentum on 2-plane and states with given winding number on dual 2-plane).

$$F^{-1} = 1 + \alpha\beta\rho^2, \quad \varphi' \equiv \varphi + \frac{1}{2}q(u+v).$$

Here $\varphi \in (0, 2\pi)$ is the periodic coordinate dual to $\tilde{\varphi}$. The models (9.1)–(9.3) are the special cases of (9.8). The theory (9.8) is conformally invariant to all orders in α' . For the purpose of demonstrating this one may ignore the difference between φ' and φ (i.e. may set $q = 0$ or consider y to be non-compact). Then (9.8) becomes equivalent to (8.5), which is the special case of the ‘generalised F -model’ (8.3), (8.4). The model (9.8) is the string theory corresponding to the $D = 4$ magnetic backgrounds (8.14)–(8.17).

One may wonder why we need to use the dual 2-plane as part of our model, i.e. why not to start with a locally flat model with $\rho^{-2}\partial\tilde{\varphi}\bar{\partial}\tilde{\varphi}$ in (9.7) replaced by $\rho^2\partial\tilde{\varphi}\bar{\partial}\tilde{\varphi}$ and constant dilaton. It turns out that in this case the resulting dual model (i) is not related to magnetic backgrounds like Melvin one, and (ii) is not exactly solvable in contrast to the one related to (9.7). To give an idea why this is so let us consider the simple case of the model with $\alpha = q = 0$. Then making the duality in $\tilde{\varphi}$ in (9.7) one finds²⁶

$$\tilde{I} = \frac{1}{\pi\alpha'} \int d^2\sigma (\partial u \bar{\partial} v + \beta\rho^2 \bar{\partial}\varphi \partial u + \partial\rho \bar{\partial}\rho + \rho^2 \partial\varphi \bar{\partial}\varphi + \mathcal{R}\phi_0). \quad (9.9)$$

Written in terms of $x = x_1 + ix_2 = \rho e^{i\varphi}$ the action becomes quadratic in x

$$\tilde{I} = \frac{1}{\pi\alpha'} \int d^2\sigma [\partial u \bar{\partial} v + \frac{1}{2}i\beta(x\bar{\partial}x^* - x^*\bar{\partial}x)\partial u + \partial x \bar{\partial}x^* + \mathcal{R}\phi_0], \quad (9.10)$$

and, as a result, is exactly solvable [51]. For example, the torus partition is explicitly computable since the integral over v constrains u to the zero (winding) mode value and thus the gaussian integral over x gives a determinant which can be expressed in terms of the Jacobi θ_1 -function. At the same time, neither the model

$$I' = \frac{1}{\pi\alpha'} \int d^2\sigma (\partial u \bar{\partial} v + \beta\bar{\partial}\tilde{\varphi} \partial u + \partial\rho \bar{\partial}\rho + \rho^2 \partial\tilde{\varphi} \bar{\partial}\tilde{\varphi} + \mathcal{R}\phi_0), \quad (9.11)$$

nor its φ -dual

$$\tilde{I}' = \frac{1}{\pi\alpha'} \int d^2\sigma [\partial u \bar{\partial} v + \beta\rho^{-2} \bar{\partial}\varphi \partial u + \partial\rho \bar{\partial}\rho + \rho^{-2} \partial\varphi \bar{\partial}\varphi + \mathcal{R}(\phi_0 - \ln\rho)], \quad (9.12)$$

become gaussian when expressed in terms of the cartesian coordinates x_1, x_2 .

9.2. Solution of the string model: quantum Hamiltonian

Relation to flat space model via formal duality and coordinate shifts makes possible to solve the classical string equations corresponding to (9.8) explicitly (the two dual models have related classical solutions), expressing the solution in terms of free fields satisfying ‘twisted’ boundary conditions [51,22]. One can then proceed with straightforward operator

²⁶ As was mentioned above, the limit of non-compact y is the plane-wave model of [37] a relation of which to the flat model by a formal $O(3,3;R)$ duality was noted in [39,92]. Our model with u, v defined in (9.4) can be interpreted as a plane wave moving in the direction of the compact internal coordinate y .

quantisation (fixing, e.g., a light-cone type gauge). Some of the resulting expressions are similar to the ones appearing in the simpler cases of open string theory in a constant magnetic field [130] or R^2/Z_N orbifold models [138].

Introducing the free field $X = X_1 + iX_2$ such that

$$L_0 = \partial_+ \rho \partial_- \rho + \rho^2 \partial_+ \hat{\varphi} \partial_- \hat{\varphi} = \partial_+ X \partial_- X^* , \quad X \equiv \rho e^{i\hat{\varphi}} , \quad (9.13)$$

$$\rho^2 = XX^* , \quad \hat{\varphi} = \frac{1}{2i} \ln \frac{X}{X^*} , \quad X = X_+(\sigma_+) + X_-(\sigma_-) , \quad \sigma_{\pm} = \tau \pm \sigma , \quad (9.14)$$

we get

$$\partial_{\pm} \tilde{\varphi} = \mp \rho^2 \partial_{\pm} \hat{\varphi} = \pm \frac{i}{2} (X^* \partial_{\pm} X - X \partial_{\pm} X^*) , \quad (9.15)$$

$$\tilde{\varphi}(\sigma, \tau) = 2\pi\alpha' [J_-(\sigma_-) - J_+(\sigma_+)] + \frac{i}{2} (X_+ X_-^* - X_+^* X_-) , \quad (9.16)$$

$$J_{\pm}(\sigma_{\pm}) \equiv \frac{i}{4\pi\alpha'} \int_0^{\sigma_{\pm}} d\sigma_{\pm} (X_{\pm} \partial_{\pm} X_{\pm}^* - X_{\pm}^* \partial_{\pm} X_{\pm}) . \quad (9.17)$$

One then finds

$$u = U_+ + U_- - \alpha\tilde{\varphi} , \quad v = V_+ + V_- - \beta\tilde{\varphi} , \quad (9.18)$$

where U_{\pm} and V_{\pm} are arbitrary functions of σ_{\pm} , and

$$x \equiv \rho e^{i\varphi} = e^{-iq(u+v)} e^{i\alpha V_- - i\beta U_+} X . \quad (9.19)$$

The closed string boundary condition $x(\sigma + \pi, \tau) = x(\sigma, \tau)$ implies that the free field $X = X_+ + X_-$ must satisfy the “twisted” condition

$$X(\sigma + \pi, \tau) = e^{i\gamma\pi} X(\sigma, \tau) , \quad X_{\pm} = e^{\pm i\gamma\sigma_{\pm}} \mathcal{X}_{\pm} , \quad \mathcal{X}_{\pm}(\sigma_{\pm} \pm \pi) = \mathcal{X}_{\pm}(\sigma_{\pm}) , \quad (9.20)$$

where $\mathcal{X}_{\pm} = \mathcal{X}_{\pm}(\sigma_{\pm})$ are free fields with standard periodic boundary conditions

$$\mathcal{X}_+ = i\sqrt{\alpha'/2} \sum_{n=-\infty}^{\infty} \tilde{a}_n \exp(-2in\sigma_+) , \quad \mathcal{X}_- = i\sqrt{\alpha'/2} \sum_{n=-\infty}^{\infty} a_n \exp(-2in\sigma_-) , \quad (9.21)$$

and γ is determined by the above relations and periodicity conditions (see below). Then

$$\tilde{\varphi}(\sigma + \pi, \tau) = \tilde{\varphi}(\sigma, \tau) - 2\pi\alpha' J , \quad J \equiv J_L + J_R , \quad J_{L,R} \equiv J_{\pm}(\pi) . \quad (9.22)$$

Since $y = \frac{1}{2}(u + v)$ is compactified on a circle of radius R ,

$$u(\sigma + \pi, \tau) = u(\sigma, \tau) + 2\pi w R , \quad v(\sigma + \pi, \tau) = v(\sigma, \tau) + 2\pi w R , \quad w = 0, \pm 1, \dots , \quad (9.23)$$

where w is a winding number. As a result,

$$U_{\pm} = \sigma_{\pm} p_{\pm}^u + U'_{\pm} , \quad V_{\pm} = \sigma_{\pm} p_{\pm}^v + V'_{\pm} , \quad (9.24)$$

$$p_{\pm}^u = \pm(wR - \alpha'\alpha J) + p_u , \quad p_{\pm}^v = \pm(wR - \alpha'\beta J) + p_v , \quad (9.25)$$

where U'_\pm and V'_\pm are single-valued functions of σ_\pm and p_u and p_v are arbitrary parameters (related to the Kaluza-Klein momentum and the energy of the string). Then the ‘twist’ parameter γ in (9.20) is given by

$$\gamma = (2q + \beta - \alpha)wR + \beta p_u + \alpha p_v . \quad (9.26)$$

Evaluating the classical stress-energy tensor on the general solution (9.14),(9.19) one finds that it takes the “free-theory” form

$$T_{\pm\pm} = \partial_\pm U_\pm \partial_\pm V_\pm + \partial_\pm X \partial_\pm X^* . \quad (9.27)$$

This is not surprising since the on-shell values of the stress-energy tensors in the two dual σ -models should be the same. It is convenient to fix the light-cone gauge, using the remaining conformal symmetry to gauge away, e.g., the non-zero-mode parts U'_\pm . Then the classical constraints $T_{--} = T_{++} = 0$ can be solved as usual and determine the remaining oscillators of V'_\pm in terms of the free fields X_\pm . The classical expressions for the Virasoro operators L_0, \tilde{L}_0 are

$$L_0 \equiv \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma T_{--} = \frac{p_-^u p_-^v}{4\alpha'} + \frac{1}{2} \sum_n \left(n + \frac{1}{2}\gamma\right)^2 a_n^* a_n , \quad (9.28)$$

$$\tilde{L}_0 \equiv \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma T_{++} = \frac{p_+^u p_+^v}{4\alpha'} + \frac{1}{2} \sum_n \left(n - \frac{1}{2}\gamma\right)^2 \tilde{a}_n^* \tilde{a}_n . \quad (9.29)$$

where $p_\pm^{u,v}$ are given by (9.25) with the angular momentum defined in (9.22),(9.17) having the following mode expansion:

$$J = J_R + J_L , \quad J_R = -\frac{1}{2} \sum_n (n + \frac{1}{2}\gamma) a_n^* a_n , \quad J_L = -\frac{1}{2} \sum_n (n - \frac{1}{2}\gamma) \tilde{a}_n^* \tilde{a}_n . \quad (9.30)$$

One can then quantize the theory by imposing the canonical commutation relations $[P_x(\sigma, \tau), x^*(\sigma', \tau)] = -i\delta(\sigma - \sigma')$, etc. As a result, p_u, p_v and the Fourier modes a_n, \tilde{a}_n will become operators in a Hilbert space. Again, the duality between (9.8) and (9.7) implies that imposing the above commutation relations is equivalent to demanding the canonical commutation relations for the fields X, X^* of the free (but globally non-trivial, cf.(9.20)) theory, $[P_X(\sigma, \tau), X^*(\sigma', \tau)] = -i\delta(\sigma - \sigma')$, etc., where $P_X(\sigma, \tau) = \frac{1}{4\pi\alpha'} \partial_\tau X$. Using (9.20), we get

$$[a_n, a_m^*] = 2(n + \frac{1}{2}\gamma)^{-1} \delta_{nm} , \quad [\tilde{a}_n, \tilde{a}_m^*] = 2(n - \frac{1}{2}\gamma)^{-1} \delta_{nm} . \quad (9.31)$$

One also finds that p_u, p_v in (9.25) and thus γ in (9.26) commute with the mode operators. The string energy and the Kaluza-Klein linear momentum operators are given by

$$E = \int_0^\pi d\sigma P_t , \quad p_y = \int_0^\pi d\sigma P_y = \frac{m}{R} , \quad m = 0, \pm 1, \pm 2, \dots , \quad (9.32)$$

$$E = \frac{1}{2\alpha'} [p_u - p_v - \alpha'(\alpha + \beta)\hat{J}] , \quad p_y = \frac{1}{2\alpha'} [p_u + p_v + \alpha'(2q + \beta - \alpha)\hat{J}] . \quad (9.33)$$

Here \hat{J} is the angular momentum operator obtained by ‘symmetrizing’ the classical expression $J = J_R + J_L$ in (9.30). Then

$$\gamma = (2q + \beta - \alpha)wR + \alpha'[(\alpha + \beta)mR^{-1} - (\alpha - \beta)E] - \frac{1}{2}\alpha'q(\alpha + \beta)\hat{J}. \quad (9.34)$$

The Virasoro operators \hat{L}_0 and $\tilde{\hat{L}}_0$ (and thus the quantum Hamiltonian $\hat{H} = \hat{L}_0 + \tilde{\hat{L}}_0$) are found by symmetrizing the mode operator products in (9.28),(9.29). In agreement with the defining relations in (9.20), the expressions for \hat{H} , \hat{J} and the commutation relations (9.31) are invariant under $\gamma \rightarrow \gamma + 2$ combined with the corresponding renaming of the mode operators $a_n \rightarrow a_{n+1}$, $\tilde{a}_n \rightarrow \tilde{a}_{n-1}$.

The sectors of states of the model can be labeled by the conserved quantum numbers: the energy E , the angular momentum \hat{J} in the x_1, x_2 plane, and the linear m/R and winding wR Kaluza-Klein momenta or “charges” (and also by momenta in additional 22 spatial dimensions which we shall add to saturate the central charge condition). As in the case of the Landau model or the open string model [130], the states with generic values of γ are “trapped” by the magnetic field. The states in the “hyperplanes” in the (m, w, E, \hat{J}) space with $|\gamma| = 2n$, $n = 0, 1, \dots$, are special: for them the translational invariance on the (x_1, x_2) -plane is restored with the zero-mode oscillators $a_0, a_0^*, \tilde{a}_0, \tilde{a}_0^*$ being replaced by the zero mode coordinate and conjugate linear momentum.

Restricting the consideration to the sector of states with $0 < \gamma < 2$ one can introduce the normalized creation and annihilation operators which will be used to define the Fock space of the model

$$[b_{n\pm}, b_{m\pm}^\dagger] = \delta_{nm}, \quad [\tilde{b}_{n\pm}, \tilde{b}_{m\pm}^\dagger] = \delta_{nm}, \quad [b_0, b_0^\dagger] = 1, \quad [\tilde{b}_0, \tilde{b}_0^\dagger] = 1, \quad (9.35)$$

where $b_{n+}^\dagger = a_{-n}\omega_-$, $b_{n-} = a_n\omega_+$, $b_0 = \frac{1}{2}\sqrt{\gamma}a_0$, $\tilde{b}_0^\dagger = \frac{1}{2}\sqrt{\gamma}\tilde{a}_0$, etc., $\omega_\pm \equiv \sqrt{\frac{1}{2}(n \pm \frac{1}{2}\gamma)}$, $n = 1, 2, \dots$. Then the final expressions for the quantum Virasoro operators and Hamiltonian take the form [22]

$$\hat{H} = \hat{L}_0 + \tilde{\hat{L}}_0 = \frac{1}{2}\alpha'(-E^2 + p_a^2 + \frac{1}{2}Q_+^2 + \frac{1}{2}Q_-^2) + N + \tilde{N} - 2c_0 \quad (9.36)$$

$$\begin{aligned} & -\alpha'[(q + \beta)Q_+ + \beta E]J_R - \alpha'[(q - \alpha)Q_- + \alpha E]J_L \\ & + \frac{1}{2}\alpha'q[(q + 2\beta)J_R^2 + (q - 2\alpha)J_L^2 + 2(q + \beta - \alpha)J_R J_L], \end{aligned}$$

$$\hat{L}_0 - \tilde{\hat{L}}_0 = N - \tilde{N} - mw. \quad (9.37)$$

Here Q_\pm are the left and right combinations of the Kaluza-Klein linear and winding momenta (which play the role of charges in the present context)

$$Q_\pm \equiv \frac{1}{\sqrt{\alpha'}}\left(\frac{m}{r} \pm wr\right), \quad r \equiv \frac{R}{\sqrt{\alpha'}}, \quad (9.38)$$

and p_a , $a = 3, \dots, 24$ are momenta corresponding to additional free spatial dimensions. c_0 is the normal ordering term²⁷

$$c_0 \equiv 1 - \frac{1}{4}\gamma(1 - \frac{1}{2}\gamma), \quad (9.39)$$

where γ was given in (9.34), and the operators N, \tilde{N} and the angular momentum operators J_L, J_R have the standard ‘free-theory’ form (a_{na}, \tilde{a}_{na} are operators corresponding to additional free spatial directions, $a = 3, \dots, 24$)

$$\begin{aligned} N &= \sum_{n=1}^{\infty} n(b_{n+}^{\dagger} b_{n+} + b_{n-}^{\dagger} b_{n-} + a_{na}^{\dagger} a_{na}), \quad \tilde{N} = \sum_{n=1}^{\infty} n(\tilde{b}_{n+}^{\dagger} \tilde{b}_{n+} + \tilde{b}_{n-}^{\dagger} \tilde{b}_{n-} + \tilde{a}_{na}^{\dagger} \tilde{a}_{na}), \\ \hat{J}_R &= -b_0^{\dagger} b_0 - \frac{1}{2} + \sum_{n=1}^{\infty} (b_{n+}^{\dagger} b_{n+} - b_{n-}^{\dagger} b_{n-}) \equiv J_R - \frac{1}{2} \rightarrow -l_R - \frac{1}{2} + S_R, \\ \hat{J}_L &= \tilde{b}_0^{\dagger} \tilde{b}_0 + \frac{1}{2} + \sum_{n=1}^{\infty} (\tilde{b}_{n+}^{\dagger} \tilde{b}_{n+} - \tilde{b}_{n-}^{\dagger} \tilde{b}_{n-}) \equiv J_L + \frac{1}{2} \rightarrow l_L + \frac{1}{2} + S_L, \\ \hat{J} &= \hat{J}_R + \hat{J}_L = J_R + J_L = J. \end{aligned} \quad (9.40)$$

The first line in (9.36) with $c_0 \rightarrow 1$ is the Hamiltonian of the free string compactified on a circle. The second line (together with $O(J)$ term from c_0) is the analogue of the gyromagnetic interaction term for a particle in a magnetic field.²⁸ Similar term is present in the Hamiltonian of conformal model describing open string in a constant magnetic field [130,131,132]

$$\hat{H}^{(open)} = L_0 = \frac{1}{2}\alpha'(-E^2 + p_a^2) + N - c_0 - \gamma J_R, \quad (9.41)$$

$$c_0 = 1 - \frac{1}{4}\gamma(1 - \frac{1}{2}\gamma), \quad \gamma \equiv \frac{2}{\pi} |\arctan(2\alpha'\pi Q_1\beta) + \arctan(2\alpha'\pi Q_2\beta)|,$$

where Q_1, Q_2 are charges at the two ends of the open string, N and \hat{J}_R have the same form as in (9.40) and β is the magnetic field, $F_{ij} = \beta\epsilon_{ij}$. The $O(J^2)$ terms in the third line of (9.36) (and in c_0) are special to closed string theory.

²⁷ The normal ordering constant is fixed by the Virasoro algebra. The free-string constant in \hat{L}_0 is shifted from 1 to $1 - \frac{1}{4}\gamma(1 - \frac{1}{2}\gamma)$ (or to $1 - \frac{1}{4}\gamma'(1 - \frac{1}{2}\gamma')$, where $\gamma' = \gamma - 2k$ and k is an integer, in the case when $2k < \gamma < 2k + 2$). This corresponds to computing the infinite sums using the generalised ζ -function regularisation. Similar result is found in the open string theory in a constant magnetic field [130] and is characteristic to the case of a free scalar field with twisted boundary conditions. This shift is also consistent with modular invariance of the partition function.

²⁸ The $O(EJ_{L,R})$ and $O(\alpha'E^2)$ terms (explicit in (9.36) and implicit in c_0 through its dependence on γ (9.34)) reflect the non-static nature of corresponding backgrounds (related also to the presence of the antisymmetric tensor). $O(\alpha'E^2)$ terms lead to a rescaling of the coefficient in front of E^2 .

The Hamiltonian (9.36) is, in general, of *fourth* order in creation and annihilation operators.²⁹ The quartic terms are absent when $q = 0$, i.e., for example, in the ‘constant magnetic field’ model (9.1), where one finds [51]

$$\begin{aligned}\hat{H}_{\alpha=q=0} &= \frac{1}{2}\alpha'(-E^2 + p_a^2 + \frac{1}{2}Q_+^2 + \frac{1}{2}Q_-^2) + N + \tilde{N} - 2c_0 \\ &\quad - \alpha'\beta(Q_+ + E)J_R, \quad \gamma = \alpha'\beta(Q_+ + E) .\end{aligned}\tag{9.42}$$

In the $a = 1$ and $a = \sqrt{3}$ Melvin models (9.2) and (9.3) we get

$$\hat{H}_{\alpha=\beta=q} = \frac{1}{2}\alpha'(-E^2 + p_a^2 + \frac{1}{2}Q_+^2 + \frac{1}{2}Q_-^2) + N + \tilde{N} - 2c_0\tag{9.43}$$

$$-2\alpha'\beta Q_+ J_R - \alpha'\beta E J - \frac{1}{2}\alpha'\beta^2 J(J - 4J_R) ,$$

$$\gamma = \alpha'\beta Q_+ - \alpha'\beta^2 J, \quad J = J_R + J_L ,$$

$$\hat{H}_{\alpha=\beta=0} = \frac{1}{2}\alpha'(-E^2 + p_a^2 + \frac{1}{2}Q_+^2 + \frac{1}{2}Q_-^2) + N + \tilde{N} - 2c_0\tag{9.44}$$

$$- \alpha'q(Q_+ J_R + Q_- J_L) + \frac{1}{2}\alpha'q^2 J^2, \quad \gamma = 2qwR .$$

The $O(\gamma^2)$ normal ordering term in c_0 in (9.36),(9.39) implies (see (9.34)) that the quantum Hamiltonian contains $O(\alpha'^2)$ term of one order higher in α' . The presence of this higher order term is consistent with current algebra approaches in the two special cases when our model becomes equivalent to a WZW or coset model: (i) the non-compact $R = \infty$ limit of the constant magnetic field model (9.1) is equivalent [51] to the E_2^c WZW model [37] for which the quantum stress tensor contains order $1/k \sim \alpha'$ correction term [39,93] (equivalent to the term appearing in (9.42) in this limit); (ii) the non-compact limit of the Melvin model (9.2) is related [134] to a special limit of the $SL(2, R) \times R/R$ gauged WZW model, or to the $E_2^c/U(1)$ coset theory, the quantum Hamiltonian of which also contains $O(1/k)$ correction term [40,93].

The analogs of the expressions (9.36)–(9.37) in the sectors with $2k < \gamma < 2k + 2$, $k = \text{integer}$, can be found in a similar way by renaming the creation and annihilation operators. The result is the same as in (9.36)–(9.37) with the replacement $\gamma \rightarrow \gamma' = \gamma - 2k$ in c_0 .

9.3. Spectrum of states

Imposing the Virasoro conditions

$$\hat{L}_0 = \tilde{\hat{L}}_0 = 0, \quad i.e., \quad \hat{H} = 0, \quad N - \tilde{N} = mw, \tag{9.45}$$

it is straightforward to compute the spectrum of these models just as for the free string theory [51,22]. Indeed, the Hamiltonian (9.36) is in a *diagonal* form since N, \tilde{N}, J_L and J_R are diagonal in Fock space.

²⁹ It is clear from our construction that (9.36) is, at the same time, also the Hamiltonian for the φ -dual theory (9.7) (the origin of the quartic terms in \hat{H} can be traced, in particular, to the presence of the $\alpha\beta\partial\tilde{\varphi}\bar{\partial}\tilde{\varphi}$ term in (9.7)).

The continuous momenta $p_{1,2}$ corresponding to the zero modes of the coordinates $x_{1,2}$ of the plane are effectively replaced by the integer eigenvalues $l_R, l_L = 0, 1, 2, \dots$ of the zero-mode parts $b_0^\dagger b_0$ and $\tilde{b}_0^\dagger \tilde{b}_0$ of \hat{J}_R and \hat{J}_L . Thus the ‘2-plane’ part of the spectrum is discrete in the $0 < \gamma < 2$ sector (but, as mentioned above, becomes continuous when $\gamma = 0$ or $\gamma = 2$).³⁰ The generic string states are thus ‘trapped’ by the flux tube. This result is consistent with a picture of a charged string moving in a magnetic field orthogonal to the plane.

The general property of the spectrum is the appearance of new tachyonic instabilities, typically associated with states with angular momentum aligned along the magnetic field. Similar instabilities are present in point-particle field theories in external magnetic fields (and may lead to a phase transition with restoration of some symmetries, see [139]). In the context of the open string theory they were observed in [130] and further investigated in [132]. The new feature of the closed string theory is the existence of states with arbitrarily large charges. Since the critical magnetic field at which a given state of a charge Q becomes tachyonic is of order of $(\alpha' Q)^{-1}$ there is an infinite number of tachyonic instabilities for any given finite value of the magnetic field.

For example, let us consider a non-winding state at zero string excitation level $S_L = S_R = N = \tilde{N} = 0$ ($w = 0$ tachyon). The eigen-values of \hat{J}_R and \hat{J}_L are $-l_R - \frac{1}{2}$ and $l_L + \frac{1}{2}$ where $l_{L,R} = 0, 1, 2, \dots$ are the Landau-level type quantum numbers. Then in the $a = 1$ Melvin model $\hat{H} = 0$ reduces to

$$M^2 \equiv E^2 - p_a^2 = -4\alpha'^{-1} + p_y^2 - 4\beta p_y \hat{J}_R + 4\beta^2 (\hat{J}_L + \hat{J}_R) \hat{J}_R - 2\alpha' \beta^2 (p_y - \beta \hat{J})^2 \quad (9.46)$$

$$= -4\alpha'^{-1} + p_y^2 + 2\beta p_y (2l_R + 1) - 2\beta^2 (l_L - l_R) (2l_R + 1) - 2\alpha' \beta^2 [p_y - \beta(l_L - l_R)]^2 ,$$

where $p_y = m/R$ and it is assumed that $0 < \gamma = 2\alpha' \beta [p_y - \beta(l_L - l_R)] < 2$. The same expression for the spectrum (up to the $O(\alpha')$ correction coming from the γ^2 term in c_0 in (9.36)) can be found [22] by directly solving (to the leading order in α') the tachyon equation

$$\alpha' [\Delta + O(\alpha')] T = 4T , \quad \Delta = -\frac{1}{\sqrt{-G} e^{-2\phi}} \partial_\mu (\sqrt{-G} e^{-2\phi} G^{\mu\nu} \partial_\nu) . \quad (9.47)$$

In the background corresponding to the $D = 5$ Melvin model (9.2)(9.47) takes the form

$$\begin{aligned} & [-\partial_t^2 + \rho^{-1} \partial_\rho (\rho \partial_\rho) + (\rho^{-2} + 2\beta^2 + \beta^4 \rho^2) \partial_\varphi^2 \\ & + (1 + \beta^2 \rho^2) \partial_y^2 - 2\beta(1 + \beta^2 \rho^2) \partial_\varphi \partial_y] T = -4\alpha'^{-1} T , \end{aligned} \quad (9.48)$$

and (9.46) is reproduced by taking $T = \exp(iEt + ip_y y + il\varphi) \tilde{T}(\rho)$, $l = l_L - l_R$. While in Sections 7,8 and above we have assumed that the σ -model is defined in the ‘leading-order scheme’, once the Hamiltonian of the corresponding CFT is known explicitly, we may also use the ‘CFT scheme’ in which the form of the tachyon equation is not modified while

³⁰ The Hamiltonian for the case of $\gamma = 0$ is obtained by adding $\frac{1}{2}\alpha'(p_1^2 + p_2^2)$ and replacing $-b_0^\dagger b_0 - \frac{1}{2}$ and $\tilde{b}_0^\dagger \tilde{b}_0 + \frac{1}{2}$ in \hat{J}_R and \hat{J}_L in (9.40) by one half of the center of mass orbital momentum $(x_1 p_2 - x_2 p_1)$.

the σ -model background fields (and thus the string σ -model action) get α' -corrections (cf. Section 2). For the $a = 1$ Melvin model one finds (cf. (9.2))

$$\begin{aligned} L'_{\alpha=\beta=q} &= -\partial t \bar{\partial} t + \partial \rho \bar{\partial} \rho + F(\rho) \rho^2 (\partial \varphi + 2\beta \partial y)(\bar{\partial} \varphi + 2\beta \bar{\partial} y) \\ &\quad + F'(\rho) \partial y \bar{\partial} y + \partial x_3 \bar{\partial} x_3 + \mathcal{R} \phi(\rho) , \\ e^{2(\phi-\phi_0)} &= [F(\rho) F'(\rho)]^{1/2} = \rho^{-1} \sqrt{-G} , \quad F' = (1 + \beta^2 \rho^2 - 2\alpha' \beta^2)^{-1} . \end{aligned} \quad (9.49)$$

Similar correspondence between the string spectrum and the solution of the tachyon equation is found also in the constant magnetic field model (9.1) where the point-particle limit of the Hamiltonian (obtained by dropping all the oscillator terms and replacing J_R by its orbital momentum eigenvalue) is

$$\hat{H}_0 = \frac{1}{2} \alpha' \left[-E^2 + p_y^2 + p_a^2 + 2(p_y + E) \beta (l_R + \frac{1}{2}) - \frac{1}{2} \alpha' \beta^2 (p_y + E)^2 \right] - 2 . \quad (9.50)$$

The effect that the magnetic field produces on the energy of a generic state is a combination of the gyromagnetic Landau-type interaction and the influence of the space-time geometry. To illustrate the presence of the new tachyonic instabilities let us consider the $\alpha = q = 0$ model and look at the states which complete the $SU(2)_R$ massless vector multiplet in the free ($\beta = 0$) theory compactified at the self-dual radius $r = 1$. The components with $S_R \neq 0$ are given by $b_{1\pm}^\dagger |0; m = w = 1\rangle$, $b_{1\pm}^\dagger |0; m = w = -1\rangle$. For them $\tilde{N} = 0$, $J_R = -l_R \pm 1$, $l_R = 0, 1, 2, \dots$, and the energy is

$$\kappa \left[E + \kappa^{-1} \beta (\hat{J}_R + \frac{1}{2} \alpha' \beta Q_+) \right]^2 = -4\alpha'^{-1} + \kappa^{-1} (Q_+ - \beta \hat{J}_R)^2 , \quad (9.51)$$

$$\kappa \equiv 1 + \frac{1}{2} \alpha' \beta^2 , \quad Q_+ = \frac{1}{\sqrt{\alpha'}} (r + r^{-1}) .$$

At the self-dual radius, $r = 1$, an infinitesimal magnetic field $\beta > 0$ makes the component with $J_R = 1$ tachyonic. This instability is the same as in the non-abelian gauge theory [139]. Away from the self-dual radius, this state has real energy for small β and becomes tachyonic at some critical magnetic field.

Instabilities caused by the linear in $\hat{J}_{L,R}$ terms in \hat{H} are present also in the $\alpha = \beta$ models, in particular, in the $a = 1$ Melvin model. There are infinitely many tachyonic charged states at higher levels. Consider, e.g., the level one state with $w = 0$, $m > 0$, $N = \tilde{N} = 1$, $l_R = l_L = 0$, $S_R = 1, S_L = -1$ (i.e. $\hat{J}_{R,L} = \pm \frac{1}{2}$) which corresponds to a ‘massless’ scalar state with a Kaluza-Klein charge. We may assume without loss of generality that $R > \sqrt{\alpha'}$ (if $R < \sqrt{\alpha'}$ a similar discussion applies with m replaced by w). Then the expression for the mass is (cf. (9.46))

$$M^2 = p_y(p_y - 2\beta - 2\alpha' \beta^2 p_y) , \quad p_y = m/R . \quad (9.52)$$

For $R \gg 1$, M^2 becomes negative when $\beta > \beta_{\text{cr}} \cong \frac{1}{2} p_y$. For these states $\gamma = 2\alpha' \beta p_y$ and thus $\gamma < 2$ if $\beta > \beta_{\text{cr}}$ and $\alpha' p_y^2 < 2$ (i.e. $R^2 > \frac{1}{2} \alpha' m^2$). The critical value of the magnetic field goes to zero as $R \rightarrow \infty$. In the noncompact $R = \infty$ theory p_y becomes a continuous parameter representing the momentum of the ‘massless’ state in the y -direction. Thus the

‘massless’ state with an infinitesimal momentum p_y becomes tachyonic for an infinitesimal value of β .

It is possible to show that the $a = \sqrt{3}$ Melvin model is stable in the non-winding ($w = 0$, i.e. $\gamma = 0$) sector, i.e. it has no new instabilities in addition to the usual flat space tachyon. For $w \neq 0$ (and $0 < \gamma < 2$) one finds

$$M^2 = 2\alpha'^{-1}(-2 + N + \tilde{N}) + (p_y - q\hat{J})^2 + [\alpha'^{-1}wR - q(\hat{J}_R - \hat{J}_L)]^2 - q^2(\hat{J}_R - \hat{J}_L)^2 - 2\alpha'^{-1}q^2w^2R^2, \quad \gamma = 2qwR, \quad (9.53)$$

and there exists a range of parameters q, R for which there is again the same linear instability as in the $a = 1$ Melvin model (9.52).

9.4. Partition function

Given the explicit expressions for the Virasoro operators (9.36),(9.37) it is straightforward to compute the partition function of this conformal model,

$$Z = \int \frac{d^2\tau}{\tau_2} \int dE \prod_{a=3}^{24} dp_a \sum_{m,w=-\infty}^{\infty} \text{Tr} \exp [2\pi i(\tau \hat{L}_0 - \bar{\tau} \hat{\tilde{L}}_0)] . \quad (9.54)$$

After the integration over the energy, momenta, Poisson resummation and introduction of two auxiliary variables $\lambda, \bar{\lambda}$ (in order to ‘split’ the $O(J^2)$ terms in $\hat{L}_0, \hat{\tilde{L}}_0$ to be able to compute the trace over the oscillators) one finds [22]

$$Z(r, \alpha, \beta, q) = c_1 \int [d^2\tau]_1 W(r, \alpha, \beta, q|\tau, \bar{\tau}) , \quad (9.55)$$

$$[d^2\tau]_1 \equiv d^2\tau \tau_2^{-14} e^{4\pi\tau_2} |f_0(e^{2\pi i\tau})|^{-48} , \quad (9.56)$$

where the integrand W is given by the sum over windings and two auxiliary ordinary integrals

$$\begin{aligned} W(r, \alpha, \beta, q|\tau, \bar{\tau}) = & r(\alpha'\alpha\beta\tau_2)^{-1} \sum_{w,w'=-\infty}^{\infty} \int d\lambda d\bar{\lambda} \\ & \times \exp \left(-\pi(\alpha'\alpha\beta\tau_2)^{-1} [\chi\tilde{\chi} + \sqrt{\alpha'}r(q+\beta)(w' - \tau w)\tilde{\chi} + \sqrt{\alpha'}r(q-\alpha)(w' - \bar{\tau}w)\chi \right. \\ & \left. + \alpha'r^2q(q+\beta-\alpha)(w' - \tau w)(w' - \bar{\tau}w)] \right) \\ & \times \exp \left[-\frac{\pi(\chi - \tilde{\chi})^2}{2\tau_2} \right] \frac{\chi\tilde{\chi}|\theta'_1(0|\tau)|^2}{\theta_1(\chi|\tau)\theta_1(\tilde{\chi}|\bar{\tau})} , \end{aligned} \quad (9.57)$$

where

$$\chi \equiv -\sqrt{\alpha'}[2\beta\lambda + qr(w' - \tau w)] , \quad \tilde{\chi} \equiv -\sqrt{\alpha'}[2\alpha\bar{\lambda} + qr(w' - \bar{\tau}w)] , \quad (9.58)$$

and λ and $\bar{\lambda}$ are independent variables having infinite integration limits. Like the measure in (9.55) W is $SL(2, Z)$ modular invariant (to show this one needs to shift w, w' and redefine $\chi, \tilde{\chi}$). $Z(r, \alpha, \beta, q)$ has several symmetry properties:

$$Z(r, \alpha, \beta, q) = Z(r, -\beta, -\alpha, q) = Z(r, -\alpha, -\beta, -q) = Z(r, \beta, \alpha, -q) . \quad (9.59)$$

It is also invariant under the duality in y direction which transforms the theory with y -period $2\pi R$ and parameters (α, β, q) into the theory with y -period $2\pi\alpha'/R$ and parameters $(q, \beta - \alpha + q, \alpha)$ or parameters $(\alpha - \beta - q, -q, -\beta)$,

$$Z(r, \alpha, \beta, q) = Z(r^{-1}, q, \beta - \alpha + q, \alpha) = Z(r^{-1}, \alpha - \beta - q, -q, -\beta) . \quad (9.60)$$

For $\alpha = q$ or $\beta = -q$ the duality relations (9.60) retain their standard ‘circle’ form

$$Z(r, \alpha, \beta, \alpha) = Z(r^{-1}, \alpha, \beta, \alpha) , \quad Z(r, \alpha, \beta, -\beta) = Z(r^{-1}, \alpha, \beta, -\beta) . \quad (9.61)$$

When $\alpha = \beta = q = 0$ the partition function Z reduces to that of the free string compactified on a circle of radius R . Taking the limit of the non-compact y -dimension ($R \rightarrow \infty$) for generic (α, β, q) one finds that Z (9.55),(9.57) reduces to the partition function of the free bosonic closed string theory.³¹

The expression for Z (9.55),(9.57) simplifies when at least one of the parameters α, β, q or $q + \beta - \alpha$ vanishes so that the integrals over $\chi, \tilde{\chi}$ can be computed explicitly. For example, in the case when either α or β is equal to zero (which includes the constant magnetic field model and $a = \sqrt{3}$ Melvin model) one finds

$$W(r, \alpha, \beta, q | \tau, \bar{\tau})|_{\alpha\beta=0} = r \sum_{w, w'=-\infty}^{\infty} \exp[-I_0(r)] \quad (9.62)$$

$$\times \exp\left[-\frac{\pi(\chi_0 - \tilde{\chi}_0)^2}{2\tau_2}\right] \frac{\chi_0 \tilde{\chi}_0 |\theta'_1(0|\tau)|^2}{\theta_1(\chi_0|\tau) \theta_1(\tilde{\chi}_0|\bar{\tau})} ,$$

$$\chi_0 = \sqrt{\alpha'}(q + \beta)r(w' - \tau w) , \quad \tilde{\chi}_0 = \sqrt{\alpha'}(q - \alpha)r(w' - \bar{\tau}w) , \quad \alpha\beta = 0 ,$$

$$I_0(r) \equiv \pi r^2 \tau_2^{-1} (w' - \tau w)(w' - \bar{\tau}w) . \quad (9.63)$$

The magnetic instability of these models (the presence of tachyons in the spectrum) is reflected in singularities (or imaginary parts) of Z . This can be seen explicitly from the behavior of the integrand for large τ_2 . The partition function has new divergences at critical values of the magnetic field parameters when the energy develops an imaginary part.

10. Concluding remarks

We have discussed several classes of classical solutions in bosonic string theory which are exact in α' . Given that string effective field equations contain terms of all orders in α' their general solution can be represented as (φ stands for a set of ‘massless’ fields, see (1.2)): $\varphi = \varphi_0 + \alpha'\varphi_1 + \alpha'^2\varphi_2 + \dots$, where φ_0 is the leading-order solution while φ_n are, in general, non-local (involving inverse Laplacians) functionals of φ_0 . It may happen that for some special φ_0 all higher order corrections are *local, covariant* functionals of φ_0 . In that case one is able to change the scheme (i.e., to make a local covariant field redefinition)

³¹ This generalizes a similar observation for the $\alpha = q = 0$ model [51]. In the limit $R = \infty$ the $\alpha = q = 0$ model (9.1) is equivalent to the model of [37] which has trivial (free) partition function [39].

so that in the new scheme φ_0 is actually an exact solution. All of the presently explicitly known exact string solutions are of that type; we do not know an example of a solution with truly non-trivial α' -dependence (the one which cannot be absorbed into a local field redefinition). In general, it seems to be an open question whether all leading-order string solutions (e.g., Schwarzschild) which can, of course, be formally deformed order by order in α' to make them satisfy the full string equations, do have extensions to *regular* (satisfying certain standard axioms) CFT's which correspond to the resulting α' -series.³²

While the form of a deformation of a leading-order solution into an exact one may not be necessary to know explicitly in the case of 'internal' string solutions describing compactified spatial dimensions (provided one knows the underlying CFT), the exact form of the background *is* important in the case of 'space-time' solutions where one is interested in large-distance interpretation of a solution. The determination of exact conformal σ -model may, in fact, be considered as a first step towards finding the corresponding CFT.

As we have seen, special chiral null models are not modified by α' corrections. This suggests that there exist well-defined CFT's corresponding to these models, though (in contrast to the magnetic flux tube models) they may be hard to solve in general (i.e., for the cases which are not, at the same time, gauged WZW models). The knowledge of these CFT's, e.g., for the fundamental string and extreme black hole solutions, would be important in order to be able to address the question of singularities. The latter are determined by the properties of test quantum string propagation in these backgrounds which are encoded in CFT. In particular, CFT determines the exact form of the center-of-mass scalar (tachyon) equation; conclusions drawn from the analysis of just the leading order form of the tachyon equation³³ may turn out to be misleading.

We have seen that some spherically symmetric solutions (e.g., (4.9), (4.11)) having singularity at the origin need to be supported by sources to satisfy the conformal invariance conditions at all points. Does that mean that they are not described by a regular CFT? A related issue is a breakdown of correspondence between solutions of β -function equations and effective string equations at points where the metric κ^{ij} in (1.2) is degenerate. For example, the fundamental string solution needs a string source if viewed as a solution of the equations following from the effective action [100,35] but the corresponding σ -model (5.1),(7.12) is automatically conformal at all points [20].

It is sometimes suggested to discriminate between 'fundamental' and 'solitonic' string solutions, depending on whether they need or not need to be supported by sources and whether the string metric is geodesically complete or not (see, e.g., [142] and refs. there). Both criteria do not seem to be unambiguous as is demonstrated by the fundamental

³² Superstring solutions represented by σ -models with (2,2) world sheet supersymmetry are examples where this does seem to happen. The corresponding β -function equations contain non-trivial α'^3 - and higher- order corrections [140] which cannot be redefined away by a local redefinition of the *metric* (otherwise one would eliminate the R^4 - term in the effective action [12] and thus change the string S-matrix). The fact that there exists a local redefinition of the Kahler potential [141] seems to indicate that the (2,2) case is special compared to the general (1,1) one. One expects that the resulting α' -deformed solution corresponds to a regular (2,2) superconformal theory.

³³ For the fundamental string background the leading-order form of this equation has a remarkably simple form of the Schrodinger equation for a charge in the Coulomb potential, suggesting that the origin $r = 0$ is not a singular point, see also [120].

string example.³⁴ In particular, the question of singularities should really be addressed at the CFT level. These issues related to CFT description of exact spherically symmetric solutions, their singularities and the role of sources need further clarification.

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³⁴ A peculiarity of the fundamental string solution is that it admits two possible interpretations. It can be considered as a vacuum solution of the effective equations (an extremal limit of a general class of charged string solutions [111]) which is valid only outside the core $r = 0$. Alternatively, it can be derived [35] as a solution corresponding to the combined action $\hat{S} = g_0^{-2}S(\varphi) + I_{str}(x; \varphi)$, $I_{str} = I_0(x) + V_i \varphi^i$, containing both the effective action for the background fields $\varphi^i = (G, B, \phi)$ (condensates of massless string modes) and the action I_{str} of a source string interacting with the background. The static source action leads to the $\delta(r)$ -term in the Laplace equation for the basic function F^{-1} (see (5.1)). Though such mixture of actions looks strange from the point of view of perturbative string theory, \hat{S} can be interpreted as describing a non-perturbative ‘thin handle’ (or ‘wormhole’) approximation to quantum string partition function [143]. Extrema of \hat{S} can thus be viewed as some non-perturbative solitonic solutions in string theory. From this point of view it is not clear if they can actually be described by a conformal field theory since what is known only is that the solutions of the tree-level effective equations $\delta S/\delta \varphi = 0$ correspond to CFT’s.

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